Appendix For "Conflicts that Leave Something to Chance"

June 13, 2022

Contents

1 Proving Lemma 1

Lemma 1 establishes, for a set of parameters, that the selected level of arming p is increasing in D's type. This Lemma is useful on two accounts. First, it is critical to Remark 1, which establishes the positive monotonic relationship between type and arming across the entire set of possible parameters. Second, the set of parameters included within Lemma 1 span several equilibria spaces. This allows me to refer to the monotonicity result within Lemma 1 at several points to make the proofs more abbreviated.

Lemma 1: Suppose $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$ and $p^D(\underline{v}_D) < p^C$. Within this region, high types select greater arming levels (i.e. $p^*(\bar{v}_D) \geq p^*(\underline{v}_D)$).[1]

Proof: Whenever $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$ and $p^D(\underline{v}_D) < p^C$, strategic play is relatively straightforward. Due to $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$, low types become willing to deter or fight rather than just arm to level p_0 and let C take the asset.[2] And, due to $p^D(\underline{v}_D) < p^C$, each type D faces their own optimization problem with their arming decision that fully determining downstream play. Here D's selected arming level will either result in them acquiescing (when the selected p is such that $p < p^D(v_D)$, fighting (when $p \in [p^D(v_D), p^C)$, or deterring ($p \geq p^C$).[3] Put another way, under these parameters , both types of D face a non-continuous, non-concave optimization

[1] Technically the set p^* is non-decreasing in private type.

[2] When $\underline{v}_D - K(p^D(\underline{v}_D)) < 0$, low types prefer setting $p = p_0$ and recieving payoff 0 to arming to level $\underline{v}_D$ and getting the good with certainty.

[3] This behavior is all part of equilibrium play as characterized below.

problem with respect to arming, where their utility function for all $p \in [p_0, p_1]$ is

$$U_D(p; v_D) = \begin{cases} 0 - K(p) & if\, p < p^D(v_D) \\ -\frac{np(1-p)}{\alpha+np(1-p)}N_D + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - \frac{c_D p(1-p)}{\alpha+np(1-p)} - K(p) & if\, p^D(v_D) \leq p < p^C \\ v_D - K(p) & if\, p^C \leq p \end{cases}$$

Because I have kept things general and cannot identify an explicit solution, for Lemma 1 to hold, I must show that in this region enough structure exists where high type D's will always select weakly lower levels of arming than low-type D's. This part of the proof will utilize the Topkis Monotonicity Theorem (Topkis 1978; Milgrom and Shannon, Econometrica 1994). For ease, I define the relevant increasing differences condition:

*Definition: Function $U_D : [p_0, p_1] \times \{\underline{v}_D, \bar{v}_D\} \to \mathbb{R}$ has **increasing differences (ID)** in (p, v_D) if, for all $p' > p$ and $v'_D > v_D$, $U_D(p', v'_D) - U_D(p, v'_D) \geq U_D(p', v_D) - U_D(p, v_D)$.*

The Topkis Monotonicity Theorem can then clarify the relationship between the set of selected arming levels $p^*(v_D) = argmax_{p \in [p_0, p_1)} U_D(p; v_D)$ and D's private value v_D. This is defined as the following:

Topkis Monotonicity Theorem: If $U_D(p; v_D)$ has increasing differences (ID) in p and v_D, then $p^*(v_D)$ is non-decreasing.

To use the Topkis Theorem, I first show that an optimal p (or set of p's) exist by demonstrating that there are no "open set" issues. In the first region of the utility function, or **Region 1** (the region where the selected $p < p^D(v_D)$), D's utility is strictly decreasing in p, meaning the optimal p for this region is p_0 (so long that $p_0 < p^D(v_D)$).[4] Next, **Region 3** (where $p^C \leq p$), D's utility is strictly decreasing, making p^C the optimal arming level. Finally, consider an analysis of a modified **Region 2**. Consider the function $V(p) = -\frac{np(1-p)}{\alpha+np(1-p)}N_D + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - \frac{c_D p(1-p)}{\alpha+np(1-p)} - K(p)$ that is optimized over the set $p^D(v_D) \leq p \leq p^C$. So long that $V$is not maximized at p^C, then there is a clearly defined optimum to $U_D(p)$ over the span of Region 2 ($p^D(v_D) \leq p < p^C$). If V is maximized at p^C, then based on the parameter assumptions, D would do strictly better setting $p = p^C$ and attaining utility $v_D - K(p^C)$.[5] Together, this means that the discontinuities between Regions will never create an open set issue, making this a well-define optimization problem with at least one solution.

Having established a non-empty set of optima exist for $U_D(p; v_D)$ as defined above, I must

[4]Consider the edge case where $p_0 = p^D(v_D)$. In this case, D's utility from selecting p_0 then acquiescing when challenged is the same as their utility from selecting p_0 then fighting when challenged. So, for the equilibrium that I consider, D will always fight.

[5]This will be the case because $\alpha p/(\alpha + np(1-p))$ is always less than 1.

show that the above utility function exhibits increasing differences (ID) in v_D and p,

It is straightforward to see that within Regions 1 and 3—in other words, for a p, p' pair such that both p fall within Region 1 (or Region 3)—(ID) holds with equality. Within Region 2, (ID) is equivalent to showing

$$\frac{\alpha}{\alpha + np'(1-p')}(p'v'_D) - \frac{\alpha}{\alpha + np(1-p)}(pv'_D) - \left(\frac{\alpha}{\alpha + np'(1-p')}(p'v_D) - \frac{\alpha}{\alpha + np(1-p)}(pv_D)\right) \geq 0$$

or

$$\left(\frac{\alpha p'}{\alpha + np'(1-p')} - \frac{\alpha p}{\alpha + np(1-p)}\right)(v'_D - v_D) \geq 0.$$

This will hold so long that

$$\frac{\alpha p'}{\alpha + np'(1-p')} - \frac{\alpha p}{\alpha + np(1-p)} \geq 0$$

or

$$\alpha^2 p' - \alpha^2 p + \alpha npp'(1-p) - \alpha npp'(1-p') \geq 0,$$

which will hold because $p' > p$.

Across regions is slightly more complicated, but note that the following properties hold:

Property (a): if $p \geq p^D(v_D)$, then $-\frac{np(1-p)}{\alpha+np(1-p)}N_D + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - \frac{c_D p(1-p)}{\alpha+np(1-p)} \geq 0$.[6]

Property (b): $\frac{\alpha p}{\alpha+np(1-p)} < 1$ and $\frac{\alpha p'}{\alpha+np'(1-p')} < 1$.[7]

Property (c): if $p \geq p^D(v_D)$, then $-\frac{np(1-p)}{\alpha+np(1-p)}N_D + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - \frac{c_D p(1-p)}{\alpha+np(1-p)}$ is increasing in p.[8]

[6] This holds based on how $p^D(v_D)$ is defined: when $p \geq p^D(v_D)$, then D is willing to fight and attain utility $-\frac{np(1-p)}{\alpha+np(1-p)}N_D + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - \frac{c_D p(1-p)}{\alpha+np(1-p)}$ over acquiesce and attain utility 0.

[7] This holds by virtue of $p \in [0,1]$.

[8] Taking first order conditions gives $\frac{d}{dp}\left(-\frac{p(1-p)}{\alpha+np(1-p)}(nN_D + c_D) + \frac{\alpha}{\alpha+np(1-p)}(pv_D)\right) = \frac{\alpha(2p-1)(c_D+nN_D)+\alpha v_D(\alpha+np^2)}{(\alpha-n(p-1)p)^2}$, or equal to $\frac{\alpha p(c_D+nN_D)+\alpha(1-p)(-c_D-nN_D)+\alpha v_D(\alpha+np^2)}{(\alpha-n(p-1)p)^2}$. The right-hand side will be positive whenever $-(1-p)(c_D + nN_D) + v_D(\alpha + np^2) \geq 0$, which will hold by Property (a). Wolfram alpha link
https://www.wolframalpha.com/input/?i=d%2Fdp+%28-%28n*N+%2Bc%29*p*%281-p%29%2F%28a+%2B+n*p*%281-p%29%29+%2B+a*p*v%2F%28a%2Bn*p*%281-p%29%29%29+

Property (d): I abuse notation and (sometimes below will) bring in the region numbers to the utility function, letting $U_D(p; v_D, 1) = -K(p)$, $U_D(p; v_D, 2) = -\frac{p(1-p)}{\alpha+np(1-p)}(nN_D + c_D) + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - K(p)$, and $U_D(p; v_D, 3) = v_D - K(p)$, regardless of p's relationship to $p^D(v_D)$ or p^C; for example, I will let $U_D(p^C; v_D, 1) = -K(p^C)$. If $p < p^D(v_D)$, then $U_D(p; v_D, 2) < U_D(p; v_D, 1)$ (because p is fixed).

To show U_D has increasing differences, I write out every case I must consider, as characterized by what Region of the utility function that the considered p or p' and v_D or v'_D put the function into. Note that there is some structure to the cases that I consider; for example, if (p', v'_D) puts the utility function into Region 2, then (p', v_D) , (p, v'_D), and (p, v_D) must fall within Region 2 or Region 1 and not Region 3, because lowering p and v_D can never shift p^C downward. And, if (p', v'_D) (or (p, v'_D)) puts the utility function into Region 3, then (p', v_D) (or (p, v_D)) must also fall within Region 3 because p^C is unchanging in v_D.

Cases	$U_D(p'; v'_D)$	$U_D(p; v'_D)$	$U_D(p'; v_D)$	$U_D(p; v_D)$
A	2	1	1	1
B	2	2	1	1
C	2	1	2	1
D	2	2	2	1
E	3	2	3	2
F	3	2	3	1
G	3	1	3	1

I now describe how increasing differences $(U_D(p', v'_D) - U_D(p, v'_D) \geq U_D(p', v_D) - U_D(p, v_D))$ occurs across all cases listed above.

Case A: $U_D(p'; v'_D) > U_D(p'; v_D)$ by property (a), and $U_D(p; v'_D) = U_D(p; v_D)$ because they are in Region 1; therefore, (ID) holds. *Case B*: by property (c) $U_D(p'; v'_D) - K(p') - (U_D(p; v'_D) - K(p)) > 0$; therefore (ID) holds. *Case C*: $U_D(p'; v'_D) > U_D(p'; v_D)$ because, in Region 2, U_D is increasing in v_D. Also, $U_D(p; v'_D) = U_D(p; v_D)$;therefore, (ID) holds. *Case D:* because Region 2 exhibits (ID), I can say $U_D(p'; v'_D) - U_D(p; v'_D) - (U_D(p'; v_D) - U_D(p; v_D, 2)) \geq 0$. By Property (d) $U_D(p; v_D, 2) < U_D(p; v_D, 1) = U_D(p; v_D)$; therefore (ID) holds. *Case E:* ID in region 2 implies $U_D(p'; v'_D, 2) - U_D(p; v'_D) - (U_D(p'; v_D, 2) - U_D(p; v_D)) \geq 0$. By property (b) $(v'_D - v_D)\left(1 - \frac{\alpha p'}{\alpha+np'(1-p')}\right) > 0$; I can add this to the left hand side and (ID) will then hold. *Case F:* (ID) in region 2 implies $U_D(p'; v'_D, 2) - U_D(p; v'_D) - (U_D(p'; v_D, 2) - U_D(p; v_D, 2)) \geq 0$. Because $(v'_D - v_D)\left(1 - \frac{\alpha p'}{\alpha+np'(1-p')}\right) > 0$, I can add this to the left hand side and get $U_D(p'; v'_D) - U_D(p; v'_D) - (U_D(p'; v_D) - U_D(p; v_D, 2)) \geq 0$. I use property (d) to say that $U_D(p; v_D, 2) < U_D(p; v_D, 1) = U_D(p; v_D)$, which will imply (ID) holds. *Case G:* $U_D(p'; v'_D) > U_D(p'; v_D)$ trivially and $U_D(p; v'_D) = U_D(p; v_D)$, meaning (ID) holds.

	Type $\underline{v}_D$ arming	Type $\bar{v}_D$ **arming**	**How is arming used? (Low-type first)**	**War with D? (Low-type first)**
Separating 1	p_0	$p^D(\bar{v}_D)$	Acquiesce, Deter	No, No
Separating 2	p_0	p^C	Acquiesce, Deter	No, No
Separating 3	p_0	$\hat{p}(\bar{v}_D)$	Acquiese, Fight	No, **Yes**
Separating 4	p_0	$\bar{p}$	Acquiese, Signal	No, No
Separating 5	$\hat{p}(\underline{v}_D)$	$\hat{p}(\bar{v}_D)$	Fight, Fight	**Yes, Yes**
Separating 6	$\hat{p}(\underline{v}_D)$	p^C	Fight, Deter	**Yes**, No
Pooling 1	p_0	p_0	Acquiese, Acquiese	No, No
Pooling 2	$p^D(\bar{v}_D)$	$p^D(\bar{v}_D)$	Bluff, Deter	No, No
Pooling 3	$\tilde{p}$	$\tilde{p}$	Bluff, Deter	No, No
Pooling 4	$p^D(\underline{v}_D)$	$p^D(\underline{v}_D)$	Deter, Deter	No, No
Pooling 5	p^C	p^C	Deter, Deter	No, No

Table 1: Equilibrium Summary. Note that implicit here is that $p_0 < p^D(\bar{v}_D)$.

Thus, increasing differences holds, and $p^*(v_D)$ is non-decreasing.

Note that in this region it is not only that $p^*(\bar{v}_D) \geq p^*(\underline{v}_D)$, but also, for example, if $\bar{v}'_D > \bar{v}_D$, $p^*(\bar{v}'_D) \geq p^*(\bar{v}_D)$. As discussed in the text, this condition does not hold throughout.

2 Equilibrium Overview

The discussion of equilibrium behavior in the main paper was written in terms of strategic behavior. This was done in an attempt to make the presentation of strategic behavior as-clear-as-possible. Here, I discuss the equilibrium characterization in terms of specific arming levels, namely, considering every unique arming pair from both types of D. For the purpose of proving the various characterizations are equilibrium behavior within a set of parameters, this is better. What does equilibrium arming behavior look like? I summarize these various arming levels in the Table below, which assumes $p_0 < p^D(\bar{v}_D)$.[9]

The way to read the table is as follows. The first column names the equilibrium, indicating whether it is pooling or separating. The second and third column specify the low-type's and high-type's arming levels. The forth column describes what the arming level accomplishes, using the terminology in the text. And the fifth column flags if war occurs or not.

[9] I express the full equilibria without this assumption in the "Characterizing and Proving the Equilibria" section. To offer one example, the arming levels in Pooling 2 without this assumption would be $p = max\left\{p_0, p^D(\bar{v}_D)\right\}$.

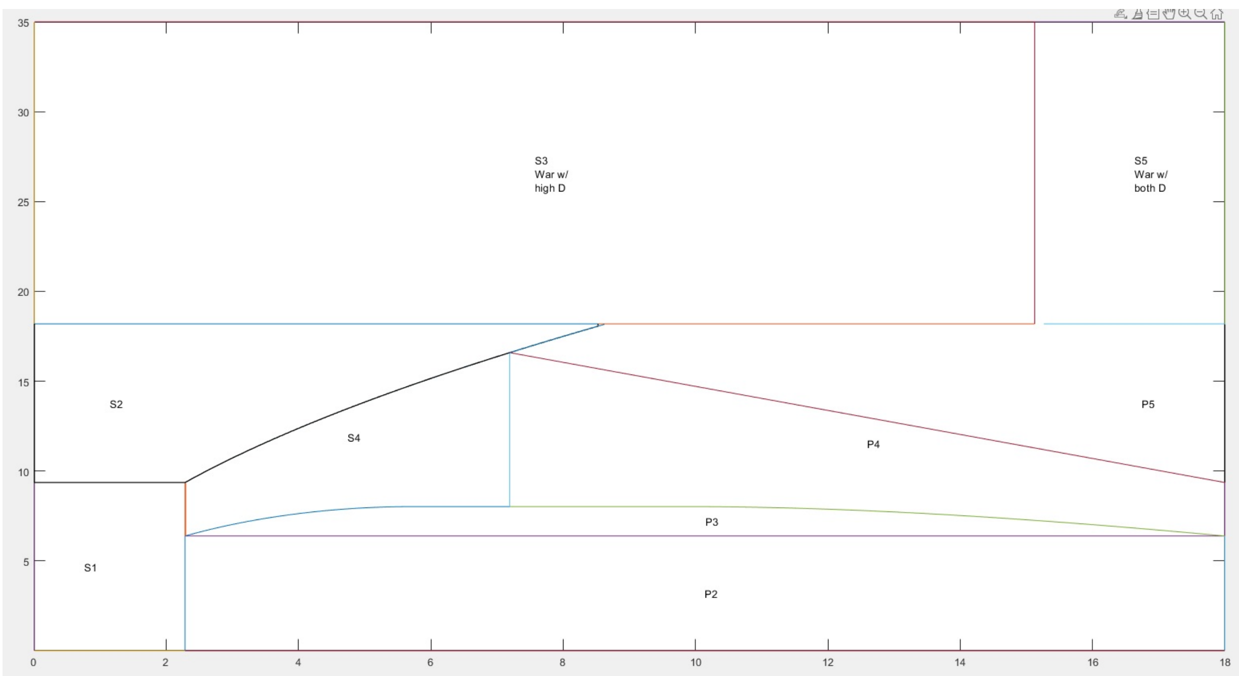

Figure 1: On the x-axis, I vary the values of $\underline{v}_D$, and on the y-axis I vary the values of v_C. "S1" is in reference to "Separating 1," and "P2" is in reference to "Pooling 2," etc. I add extra text to describe where war happens. The cost function is $k * (p^* - p_0)^2$. The non-illustrated parameter values are $p_0 = 0.001$, $p_1 = 0.999$, $c_D = 3$, $c_C = 1$, $N_C = 20$, $N_D = 40$, $\alpha = 0.1$, $n = 0.01$, $k = 15$, $\pi = 0.8$, $\bar{v}_D = 25$. Note that Pooling 1 isn't visualized; because high-type D's going to war gives these D's a better payoff than selecting p_0 and acquiescing, this equilibrium space is ruled out (if war were more expensive, P1 would exist roughly where S3 and S5 is).

To give a sense of what the game looks like, see Figure 1 displays the equilibrium for various parameters while allowing $\underline{v}_D$ and v_C to vary. $\underline{v}_D$ increases along the x-axis, and v_C increases along the y-axis. Note that these are the same parameters as Figure 1 in the main text, which allows for easy comparison between the labeling here and the labeling in the text. For example, the Separating 1 and Separating 2 equilibria form the "Deter-Acquiesce" equilibrium space, Pooling 2 and Pooling 3 form the "Deter-Bluff" equilibrium space, etc.

To give a sense of how the game plays out, recall that high-type D's are always willing to arm to level $p^D(\bar{v}_D)$ if this results in C not challenging by the parameter assumptions. In the bottom-left corner of Figure 1 (Separating 1), low-type D's care very little about the asset (low $\underline{v}_D$). In the Separating 1 parameter space, low-type D's are unwilling to arm to the level where they would imitate high-type D's, even if it led to them attaining the asset. In this parameter space, C will never challenge upon observing $p = p^D(\bar{v}_D)$ because C knows only high-types of D would be wiling to make this investment, and high-types would always fight after selecting this investment level. Also here, C will always challenge upon observing $p = p_0$, because only low-types make the low investment and C knows that if they challenge, then they will attain the asset.

Moving to the right, low-type D's care more about the asset. Within Pooling 2, low-type D's are willing to select arming level $p = p^D(\bar{v}_D)$ if it results in them attaining the asset (i.e. C not challenging).[10] That being said, if low-type D's select arming level $p = p^D(\bar{v}_D)$ and were challenged, they would not fight because $p^D(\bar{v}_D) < p^D(\underline{v}_D)$; however, in this range, because C cares so little about the asset, C is unwilling to challenge at arming level $p^D(\bar{v}_D)$ even though C knows that by challenging all low-type D's would drop out. Moving up to Pooling 3, the logic is the same, only D's must pool on a slightly higher level of arming $p = \tilde{p}$ to deter C from challenging even though C knows low-type D's would drop out if challenged.

Moving up from Pooling 3 into the Separating 4 and Pooling 4 regions, C cares more about the asset, but is still unwilling to challenge if C knew that D would fight in response. Within this range of parameters, no arming level exists where (a) high-type D's would fight when challenged, (b) low-type D's would acquiesce when challenged, and (c) C would be deterred from challenging conditional on D's behavior as characterized in Pooling 2 and 3. When low-type D's do not value the asset enough—as characterized by $v_D - k(p^D(\underline{v}_D)) < 0$)—a separating equilibrium (Separating 4) exists where high-type D's select an arming level that will insure low-type D's will not mimic them ($p = \bar{p}$), and low-type D's will select the lowest arming level $p = p_0$. In response, C would never challenge when observing $p = \bar{p}$, and would always challenge when observing $p = p_0$. When low-type D's value the asset more (Pooling 4)—as characterized by $v_D - k(p^D(\underline{v}_D)) \geq 0$—low-type D's attain a positive utility from arming to level $p = p^D(\underline{v}_D)$

[10] This holds because $\underline{v}_D - k(\bar{p}_D) \geq 0$ in this range.

and deterring C from challenging.

Moving up again, when $p^D(\bar{v}_D) < p^C$ (to Separating 2), then the level of arming that would convince a high-type D to fight after being challenged is less than the level of arming that would deter C from challenging conditional on D fighting with certainty. Thus, within this range, D must select a level of arming that exceeds $p^D(\bar{v}_D)$ to deter C. This level of arming will be p^C, the level that would make C not challenge (Separating 2). In the range of $\underline{v}_D$ values where $\bar{v}_D - k(p^D(\underline{v}_D)) \geq 0$, low-type D's begin caring enough and could select Pooling 5, where they choose p^C in order to deter C.

Finally, in the top region of the graph, high-type D's are no longer willing to arm to level p^C to deter C, or it becomes impossible with $p^C > p_1$. Under these parameters, then high-type D's will either select $p = p_0$ and acquiesce when challenged (Pooling 1), or will select $p = \hat{p}(\bar{v}_D)$ and fight when challenged (Separating 3), depending on which offers D a greater utility. When high-type D's optimally select $p = p_0$, low-type D's will always match high-type D's play and select $p = p_0$. When high-type D's optimally select $p = \hat{p}(\bar{v}_D)$, low-type D's will either select $p = p_0$ (when $\underline{v}_D$ is low, Separating 3), or will select $p = \hat{p}(\underline{v}_D)$ and will fight when challenged (Separating 5).[11]

3 Characterizing and Proving the Equilibrium

Here I fully characterize every equilibrium, and prove it's existence within the given parameter set.

3.1 Separating Equilibrium 1:

- This equilibrium occurs when $\underline{v}_D - K(p^D(\bar{v}_D)) \leq 0$, $p^D(\bar{v}_D) \geq p^C$,[12] $p^D(\bar{v}_D) > p_0$.
- Type $\bar{v}_D$ selects $p^* = p^D(\bar{v}_D)$, and type $\underline{v}_D$ selects $p^* = p_0$.
- C will challenge for all $p < p^D(\bar{v}_D)$ and will not challenge for all $p \geq p^D(\bar{v}_D)$.
- For this equilibrium and all other equilibria listed below (i.e. Separating 2, Separating 3, etc), each type of D would escalate if $p \geq p^D(v_D)$ and would not otherwise. Type $\underline{v}_D$ (who is challenged) will not escalate. Type $\bar{v}_D$ (who is not challenged) would escalate if challenged.

[11] Note: Separating 5 may not actually be separating when low and high type D's optimally select the same arming level when fighting. For example, this can occur when $p^C > p_1$ and both low and high types arm to level $p = p_1$.

[12] Recall that I assume (by the "Parameter Assumptions") $\bar{v}_D - K(p^D(\bar{v}_D)) > 0$, and $p^1 > p^D(\bar{v}_D)$.

- C's Beliefs: If $p < p^D(\bar{v}_D)$, then C believes D is low-type with probability 1. If $p \geq p^D(\bar{v}_D)$, then C believes D is high-type with probability 1.

- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(p^D(\bar{v}_D))$, type $\underline{v}_D$ attains 0.

Proof of Equilibrium:
For type $\underline{v}_D$: Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. For any $p \in [p^D(\bar{v}_D), p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^D(\bar{v}_D)$ dominates all other arming levels. Due to $\underline{v}_D - K(p^D(\bar{v}_D)) \leq 0$, low-type prefer selecting $p = p_0$ to $p = p^D(\bar{v}_D)$.

For type $\bar{v}_D$: Because after selecting $p \in [p_0, p^D(\bar{v}_D))$ it would be optimal for D to acquiesce rather than fight, [13] following the logic above, high types choose between p_0 and $p^D(\bar{v}_D)$. From the Parameter Assumptions stated in the paper $\bar{v}_D - K(p^D(\bar{v}_D)) > 0$, meaning high types prefer selecting $p = p^D(\bar{v}_D)$ to $p = p_0$.

For C: For $p \in [p_0, p^D(\bar{v}_D))$, C believes D is a low-type and would acquiesce if challenged, which gives C a strictly positive payoff for challenging. For $p \in [p^D(\bar{v}_D), p_1]$, the selected $p \geq p^C$, meaning C does weakly better not challenging.

3.2 Separating Equilibrium 2:

- This equilibrium occurs when
  (a) $p^D(\underline{v}_D) < p^C$, $max\{0, U_D(\hat{p}(\bar{v}_D))\} \leq \bar{v}_D - K(p^C)$, $0 \geq \underline{v}_D - K(p^C)$, $0 > U_D(\hat{p}(\underline{v}_D))$, $p^C \leq p_1$, [14] or
  (b) $p^D(\underline{v}_D) \geq p^C$,$p^D(\bar{v}_D) < p^C$, $max\{0, U_D(\hat{p}(\bar{v}_D))\} \leq \bar{v}_D - K(p^C)$, $0 \geq \underline{v}_D - K(p^C)$, $p^C \leq p_1$,[15]

- Type $\bar{v}_D$ selects $p^* = p^C$, and type $\underline{v}_D$ selects $p^* = p_0$.

- C will challenge for all $p < p^C$ and will not challenge for all $p \geq p^C$.

- Type $\underline{v}_D$ (who is challenged) will not escalate. Type $\bar{v}_D$ (who is not challenged) would escalate if challenged.

- C's Beliefs: If $p < p^C$, then C believes D is low-type with probability 1. If $p \geq p^C$, then C believes D is high-type with probability 1.

[13] This follows from the definition of $p^D(\bar{v}_D)$.

[14] No discussion of $p^D(\bar{v}_D)$ is required here. The condition $p^D(\underline{v}_D) < p^C$ implies $p^D(\bar{v}_D) < p^C$, meaning high type D's must arm to p^C to deter C. No discussion of $\bar{p}$ is reqired. I don't need to include the $p^C > p_0$, it's implicit due to $0 \geq \underline{v}_D - K(p^C)$.

[15] I don't need $p^C \geq \bar{p}$, this is contained within the $0 \geq \underline{v}_D - K(p^C)$ condition. I don'tneed $p^C > p_0$, the $0 \geq \underline{v}_D - K(p^C)$ condition implies it.

- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(p^C)$, type $\underline{v}_D$ attains 0.

Proof of Equilibrium:

For type $\underline{v}_D$:

Case (a). Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility is strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\underline{v}_D)$ weakly dominates all other arming levels.[16] Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $0 \geq \underline{v}_D - K(p^C)$ and $0 > U_D(\hat{p}(\underline{v}_d))$, implying that D prefers p_0 to fighting or deterring.

Case (b). Within the range $p \in [p_0, p^C)$, C will challenge and D will acquiesce, making D's utility is strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p_C, p_1]$, C will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case $0 \geq \underline{v}_D - K(p^C)$, implying that D prefers setting $p = p_0$ to deterring.

For type $\bar{v}_D$: Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p^C)$, C will challenge and D will fight; thus, in this range $p = \hat{p}(\bar{v}_D)$ weakly dominates all other arming levels.[17] Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $max\{0, U_D(\hat{p}(\bar{v}_d))\} \leq \bar{v}_D - K(p^C)$, implying that D prefers setting $p = p^C$ and deterring to fighting or acquiescing.

For C: For $p \in [p_0, p^C)$, C believes D is a low-type and would acquiesce (when $p < p^D(\underline{v}_D)$) or fight (when $p \geq p^D(\underline{v}_D)$) if challenged; regardless, C attains a strictly positive payoff for challenging (based on the p^C condition). For $p \in [p^C, p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (based on the p^C condition).

3.3 Separating Equilibrium 3:

- This equilibrium occurs when
  (a) $p^C \leq p_1$, $p^D(\bar{v}_D) < p^C$,[18] $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$, $U_D(\hat{p}(\bar{v}_D)) \geq 0$, and, when the set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p^C]$ is non-empty, $0 > U_D(\hat{p}(\underline{v}_D))$,[19] or

[16] If $\hat{p}(\underline{v}_D) = p^C$, then D would optimally select p^C and deter. This will prevent any open set issues.

[17] If $\hat{p}(\bar{v}_D) = p^C$, then D would optimally select p^C and deter. This prevents open set issues.

[18] I don't need to say that $p^C > p_0$; the condition $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$ implies this.

[19] These conditions may seem incomplete: see the proof of equilibrium below.

(b) $p^C > p_1$, $0 \leq U_D(\hat{p}(\bar{v}_d))$, and $0 > U_D(\hat{p}(\underline{v}_D))$[20]

- Type $\bar{v}_D$ selects $p = \hat{p}(\bar{v}_d)$, and type $\underline{v}_D$ selects $p = p_0$.
- C will challenge for all $p < p^C$, and will not challenge for all $p \geq p^C$. Note in Case (b), C always challenges.
- Type $\underline{v}_D$ (who is challenged) will not escalate. Type $\bar{v}_D$ (who is challenged) will escalate.
- C's Beliefs: If $p < \hat{p}(\bar{v}_d)$, then C believes D is low-type with probability 1. If $p \geq \hat{p}(\bar{v}_d)$, then C believes D is high-type with probability 1.
- Payoffs: Type $\bar{v}_D$ attains $U_D(\hat{p}(\bar{v}_D))$, type $\underline{v}_D$ attains 0.

Proof of Equilibrium:
For type $\underline{v}_D$:
Case (a),
Case (a.1). In addition to the equilibrium conditions on case (a), assume $p^C > p^D(\underline{v}_D)$ and $\underline{v}_D - K(p^D(\underline{v}_D)) < 0$. Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\underline{v}_D)$ weakly dominates all other arming levels.[21] Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $0 > U_D(\hat{p}(\underline{v}_D))$, implying that D prefers p_0 to fighting. Additionally, I can use the conditions of this sub-case $p^C > p^D(\underline{v}_D)$ and $\underline{v}_D - K(p^D(\underline{v}_D)) < 0$, which together imply $\underline{v}_D - K(p^C) < 0$, or that D prefers setting $p = p_0$ and acquiescing to setting $p = p^C$ and deterring.

Case (a.2). In addition to the equilibrium conditions on case (a), assume $p^C > p^D(\underline{v}_D)$ and $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$. This proof is identical up to the point before demonstrating that D prefers setting $p = p_0$ and acquiescing to setting $p = p^C$ and deterring. As I showed in Lemma 1, within the parameter set where $p^C > p^D(\underline{v}_D)$ and $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$, D's selected p^* is non-decreasing in v_D. Thus, because high types optimally fight (as I discuss below), low-types would never prefer setting $p = p^C$ and deterring. And, because $0 > U_D(\hat{p}(\underline{v}_D))$, low types still prefer acquiescing.

Case (a.3) In addition to the equilibrium conditions on case (a), assume $p^D(\underline{v}_D) \geq p^C$. Within the range $p \in [p_0, p^C)$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility decreasing in p. Thus, in this range, $p = p^C$

[20] The set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p_1]$ is never empty.
[21] If $\hat{p}(\underline{v}_D) = p^C$, then D would optimally select p^C and deter. This will prevent any open set issues.

dominates all other arming levels. I can demonstrate that low-type D's always prefer setting $p = p_0$ to $p = p^C$. Starting with $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$, which is given by the conditions of the case, I use the definition of $U_D(\hat{p}(\bar{v}_D))$ to say

$$-\frac{\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}(nN_D + c_D) + \frac{\alpha\hat{p}(\bar{v}_D)}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}\bar{v}_D - K(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C).$$

Because $\frac{\alpha\hat{p}(\bar{v}_D)}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}(\bar{v}_D - \underline{v}_D) < \bar{v}_D - \underline{v}_D$, I can say

$$-\frac{\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}(nN_D + c_D) + \frac{\alpha\hat{p}(\bar{v}_D)}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}\underline{v}_D - K(\hat{p}(\bar{v}_D)) > \underline{v}_D - K(p^C).$$

As how $\hat{p}(\bar{v}_D)$ is defined, it must be that $\hat{p}(\bar{v}_D) \leq p^C$, implying (by the conditions of case a.3 and transitivity) $\hat{p}(\bar{v}_D) \leq p^D(\underline{v}_D)$. Thus, from how $p^D(\underline{v}_D)$ is defined, the following expression involving $\hat{p}(\bar{v}_D)$ must hold: $-\frac{\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}(nN_D + c_D) + \frac{\alpha\hat{p}(\bar{v}_D)}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}\underline{v}_D \leq 0$. This in turn implies

$$0 - K(\hat{p}(\bar{v}_D)) > \underline{v}_D - K(p^C),$$

or $0 > \underline{v}_D - K(p^C)$. Thus, D prefers setting $p = p_0$ and acquiescing to $p = p^C$ and deterring.

Case b.
Case (b.1) In addition to the equilibrium conditions on case (b), assume $p_1 \geq p^D(\underline{v}_D)$. Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p_1]$ C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\underline{v}_D)$ weakly dominates all other arming levels. By the conditions of the case, $0 > U_D(\hat{p}(\underline{v}_d))$, implying that D prefers setting $p = p_0$ to fighting.

Case (b.2)In addition to the equilibrium conditions on case (b), assume $p_1 < p^D(\underline{v}_D)$. Within the range $p \in [p_0, p_1]$, C will challenge and D will acquiesce, making D's utility is strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels.

For type $\bar{v}_D$:
Case (a) Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\bar{v}_D)$ weakly dominates all other arming levels.[22] Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$ and

[22]If $\hat{p}(\bar{v}_D) = p^C$, then D would optimally select p^C and deter. This will prevent any open set issues.

$U_D(\hat{p}(\bar{v}_D)) \geq 0$, implying that D prefers selecting $p = \hat{p}(\bar{v}_D)$ and fighting.

Case (b) Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p_1]$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\bar{v}_D)$ weakly dominates all other arming levels. By the conditions of the case $U_D(\hat{p}(\bar{v}_D)) \geq 0$, D prefers selecting $p = \hat{p}(\bar{v}_D)$ and fighting.

For C: For $p \in [p_0, \hat{p}(\bar{v}_d))$, C believes D is a low-type and would acquiesce if challenged, which gives C a strictly positive payoff for challenging (based on $\hat{p}(\bar{v}_d) < p^C$).

Case (a) For $p \in [\hat{p}(\bar{v}_d), p^C)$, C believes that D is a high type and would fight if challenged, which gives C a weakly positive payoff for challenging (based on the p^C condition). For $p \in [p^C, p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (based on the p^C condition).

Case (b) For $p \in [\hat{p}(\bar{v}_d), p_1)$, C believes that D is a high type and would fight if challenged, which gives C a weakly positive payoff for challenging (based on the $p_1 < p^C$ condition).

3.4 Separating Equilibrium 4

- This equilibrium occurs when
  (a) $p^C \leq p^D(\bar{v}_D)$, $\underline{v}_D - K(p^D(\bar{v}_D)) > 0$, $\underline{v}_D - K\left(p^D(\underline{v}_D)\right) \leq 0$, and, if $\tilde{P}$ is non-empty, $\underline{v}_D - K(\tilde{p}) \leq 0$
  (b) $p^C > p^D(\bar{v}_D)$, $\underline{v}_D - K(p^C) > 0$, $\underline{v}_D - K\left(p^D(\underline{v}_D)\right) \leq 0$, and, if $\tilde{P}$ is non-empty, $\underline{v}_D - K(\tilde{p}) \leq 0$

- Type $\bar{v}_D$ selects $p = \bar{p}$, and type $\underline{v}_D$ selects $p = p_0$.

- C will challenge for all $p < \bar{p}$, and will not challenge for all $p \geq \bar{p}$.

- Type $\underline{v}_D$ (who is challenged) will not escalate. Type $\bar{v}_D$ is not challenged..

- C's Beliefs: If $p < \bar{p}$, then C believes D is low-type with probability 1. If $p \geq \bar{p}$, then C believes D is high-type with probability 1.

- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(\bar{p})$, type $\underline{v}_D$ attains 0.

Proof of Equilibrium:[23]
For type $\underline{v}_D$: Within the range $p \in [p_0, \bar{p})$, C will challenge and D will acquiesce, making D's

[23] This one looks like it shouldn't be like this, but the proofs of these two cases are the same. Because $\underline{v}_D - K\left(p^D(\underline{v}_D)\right) \leq 0$, in both cases I can say that $p^D(\underline{v}_D) > p^C$. So what matters here is the value where low type D's drop out, so long that the p is above $p^D(\bar{v}_D)$, and this will hold in both cases.

utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. For any $p \in [\bar{p}, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = \bar{p}$ dominates all other arming levels. By it's definition, $\underline{v}_D - K(\bar{p}) = 0$, meaning low-type D's weakly prefer selecting $p = p_0$ and acquiescing to $p = \bar{p}$ and bluffing.

For type $\bar{v}_D$: Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), \bar{p})$, C will challenge and D will fight. Thus, in this range, there exists some arming level or set of arming levels that dominates all others.[24] Within the range $p \in [\bar{p}, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = \bar{p}$ dominates all other arming levels. Because $\underline{v}_D - K(\bar{p}) = 0$, it must be that $\bar{v}_D - K(\bar{p}) > 0$, meaning D prefers arming to $p = \bar{p}$ to setting $p = p_0$. To demonstrate that D prefers setting $p = \bar{p}$ to fighting, I must first define the value $\dot{p}$ as type $\bar{v}_D$'s optimal arming level conditional on the high type looking to fight, or

$$\dot{p} \in argmax_{p \in [p^D(\bar{v}_D), \bar{p}]} \left\{ -\frac{p(1-p)}{\alpha + np(1-p)} (nN_D + c_D) + \frac{\alpha p}{\alpha + np(1-p)} \bar{v}_D - K(p) \right\}.$$

Note that because $\underline{v}_D - K(\bar{p}) = 0$ and $\underline{v}_D - K(p^D(\underline{v}_D)) \leq 0$, it must be that $\bar{p} \leq p^D(\underline{v}_D)$, meaning $\dot{p} \leq p^D(\underline{v}_D)$.

I start with a condition which follows from how $\bar{p}$ is defined:

$$\underline{v}_D - K(\bar{p}) = 0.$$

Using that $\dot{p} \leq p^D(\underline{v}_D)$, it must be that

$$\underline{v}_D - K(\bar{p}) \geq -\frac{\dot{p}(1-\dot{p})}{\alpha + n\dot{p}(1-\dot{p})} (nN_D + c_D) + \frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \underline{v}_D - K(\dot{p}).$$

Because $\bar{v}_D - \underline{v}_D > (\bar{v}_D - \underline{v}_D) \left(\frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \right)$, I can say

$$\bar{v}_D - K(\bar{p}) \geq -\frac{\dot{p}(1-\dot{p})}{\alpha + n\dot{p}(1-\dot{p})} (nN_D + c_D) + \frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \bar{v}_D - K(\dot{p}),$$

which implies that D always prefers setting $p = \bar{p}$ and deterring to selecting $p = \dot{p}$ and fighting.

For C: For $p \in [p_0, \bar{p})$, C believes D is a low-type and would acquiesce if challenged, which

[24] Note that this will not be $p = \hat{p}(\bar{v}_D)$ because the utility function is optimized over a different domain. Also, following the rationale discussed in prior footnotes, there will not be open set issues here.

gives C a strictly positive payoff. For $p \in [\bar{p}, p_1]$, C believes D is a high-type and would fight if challenged. For both cases, $\bar{p} \geq p^C$,[25] meaning C would prefer to acquiesce rather than fight with arming level $\bar{p}$.

3.5 Separating Equilibrium 5:

- This equilibrium occurs when
  (a) $p^C \leq p_1$, $max\left\{p^D(\underline{v}_D), p_0\right\} < p^C$, $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$, and $U_D(\hat{p}(\underline{v}_D)) \geq 0$, or[26]
  (b) $p^C > p_1$ and $0 \leq U_D(\hat{p}(\underline{v}_D))$.[27]
- Type $\bar{v}_D$ selects $p = \hat{p}(\bar{v}_D)$, and type $\underline{v}_D$ selects $p = \hat{p}(\underline{v}_D)$.
- C will challenge for all $p < p^C$, and will not challenge for all $p \geq p^C$.
- Both types will escalate when challenged.
- C's Beliefs: If $p < \hat{p}(\bar{v}_D)$, then C believes D is low-type with probability 1. If $p \geq \hat{p}(\bar{v}_D)$, then C believes D is high-type with probability 1.
- Payoffs: Type $\bar{v}_D$ attains $U_D(\hat{p}(\bar{v}_D))$, type $\underline{v}_D$ attains $U_D(\hat{p}(\underline{v}_D))$.

Proof of Equilibrium
Case (a).
For type $\underline{v}_D$: Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\underline{v}_D)$ weakly dominates all other arming levels. Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $U_D(\hat{p}(\underline{v}_D)) \geq 0$, implying that D prefers setting $p = \hat{p}(\underline{v}_D)$ and fighting to $p = p_0$ and acquiescing. The remainder relies on utilizing Lemma 1. I must show that the above conditions imply $\underline{v}_D - K(p^D(\underline{v}_D)) > 0$, which is needed for Lemma 1 to apply. Using $U_D(\hat{p}(\underline{v}_D)) \geq 0$, I can say

$$-\frac{\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}{\alpha + n\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}(nN_D + c_D) + \frac{\alpha\hat{p}(\underline{v}_D)}{\alpha + n\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}\underline{v}_D - K(\hat{p}(\underline{v}_D)) \geq 0,$$

which implies

$$\underline{v}_D - K(\hat{p}(\underline{v}_D)) > 0.$$

[25]In case (a), this follows from how $\bar{p}$ is defined and $p^C \leq p^D(\bar{v}_D)$ and $\underline{v}_D - K(p^D(\bar{v}_D)) > 0$. In case (b), this follows from how $\bar{p}$ is defined and $\underline{v}_D - K(p^C) > 0$.

[26]Note that based on Lemma 1, I know high-type D's would never play p_0, so I don't need to define $U_D(\hat{p}(\bar{v}_D)) \geq 0$.

[27]Recall $p^D(\underline{v}_D) \leq p_1$. Also, based on Lemma 1, high-type D's would never play p_0, so I don't need to define $U_D(\hat{p}(\bar{v}_D)) \geq 0$.

Because (by definition) $\hat{p}(\underline{v}_D) \in [p^D(\underline{v}_D), p^C]$, I can say

$$\underline{v}_D - K(p^D(\underline{v}_D)) > 0.$$

Thus, Lemma 1 can apply here. Because high types prefer setting $p = \hat{p}(\bar{v}_D)$ and fighting to setting $p = p^C$ and deterring, low-types will never set $p = p^C$.

For type $\bar{v}_D$: Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p^C)$, C will challenge and D will fight; thus, in this range $p = \hat{p}(\bar{v}_D)$ weakly dominates all other arming levels.[28] Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$ and $U_D(\hat{p}(\bar{v}_D)) \geq 0$ implying that D prefers selecting $\hat{p}(\bar{v}_D)$ and fighting to deterring or acquiescing.

For C: For $p \in [p_0, \hat{p}(\bar{v}_d))$, C believes D is a low-type and would acquiesce or fight if challenged, which gives C a strictly positive payoff for challenging (based on $\hat{p}(\bar{v}_d) < p^C$). For $p \in [\hat{p}(\bar{v}_d), p^C)$, C believes that D is a high type and would fight if challenged, which gives C a weakly positive payoff for challenging (based on the p^C condition). For $p \in [p^C, p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (based on the p^C condition).

Case (b). The proof is nearly identical, other than D can no longer select some $p \geq p^C$ and deter C.

3.6 Separating Equilibrium 6:

- This equilibrium occurs when $p^D(\underline{v}_D) < p^C$, $p^C \leq p_1$, $\bar{v}_D - K(p^C) \geq U_D(\hat{p}(\bar{v}_D))$, $U_D(\hat{p}(\underline{v}_D)) \geq 0$ and $U_D(\hat{p}(\underline{v}_D)) > \underline{v}_D - K(p^C)$[29]
- Type $\bar{v}_D$ selects $p = p^C$, and type $\underline{v}_D$ selects $p = \hat{p}(\underline{v}_D)$.
- C will challenge for all $p < p^C$, and will not challenge for all $p \geq p^C$.
- When challenged, low-types will escalate
- C's Beliefs: If $p < p^C$, then C believes D is low-type with probability 1. If $p \geq p^C$, then C believes D is high-type with probability 1.
- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(p^C)$, type $\underline{v}_D$ attains $U_D(\hat{p}(\underline{v}_D))$.

[28] If $\hat{p}(\bar{v}_D) = p^C$, then D would optimally select p^C and deter. This will prevent any open set issues.

[29] This last condition inplies that $p_0 < p^C$.

Proof of equilibrium:

For type $\underline{v}_D$: Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\underline{v}_D)$ weakly dominates all other arming levels. Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $U_D(\hat{p}(\underline{v}_D)) \geq 0$ and $U_D(\hat{p}(\underline{v}_D)) > \underline{v}_D - K(p^C)$, implying that D prefers setting $p = \hat{p}(\underline{v}_D)$ and fighting to $p = p_0$ and acquiescing or p^C and deterring.

For type $\bar{v}_D$: Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p^C)$, C will challenge and D will fight; thus, in this range $p = \hat{p}(\bar{v}_D)$ weakly dominates all other arming levels.[30] Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $\bar{v}_D - K(p^C) \geq U_D(\hat{p}(\bar{v}_D))$, implying that D prefers selecting $p = p^C$ and deterring to selecting $p = \hat{p}(\bar{v}_D)$ and fighting. And, as shown in the discussion of the Separating 5 equilibrium, $U_D(\hat{p}(\underline{v}_D)) \geq 0$ implies that $\underline{v}_D - K(p^D(\underline{v}_D)) > 0$. This means that Separating 6 falls within conditions for Lemma 1. Thus, D will never select $p = p_0$ and acquiesce because low types select $p = \hat{p}(\underline{v}_D)$.

For C: For $p \in [p_0, p^C)$, C believes D is a low-type and would either acquiesce if challenged or fight when challenged: both give C a strictly positive payoff for challenging. For $p \in [p^C, p^1)$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (based on the p^C condition).

3.7 Pooling Equilibrium 1:

- This equilibrium occurs when
  (a) $p^C \leq p_1$, $p^D(\bar{v}_D) < p^C$,[31] $p_0 < p^D(\bar{v}_D)$, $0 > \bar{U}_D(\hat{p}(\bar{v}_d))$, $0 > \bar{v}_D - K(p^C)$ or
  (b) $p^C > p_1$, $p_0 < p^D(\bar{v}_D)$, $0 > \bar{U}_D(\hat{p}(\bar{v}_D))$
- Type $\bar{v}_D$ selects $p = p_0$, and type $\underline{v}_D$ selects $p = p_0$.
- C will challenge for all $p < p^C$, and will not challenge for all $p \geq p^C$.
- Neither type will escalate when challenged.
- C's Beliefs: If $p = p_0$, then C believes D is low-type with probability $1 - \pi$ and high-type

[30] If $\hat{p}(\bar{v}_D) = p^C$, then D would optimally select p^C and deter. This will prevent any open set issues.

[31] You need this condition based on parameter assumptions: otherwise, type $\bar{v}_D$ would be willing to arm to $p^D(\bar{v}_D)$ and this would deter C (or create some other strategic play).

with probability π. If $p \neq p_0$ and $p < p^C$, then C believes D is low-type with probability 1. If $p \geq p^C$, then C believes D is high-type with probability 1.

- Payoffs: Type $\bar{v}_D$ attains 0, type $\underline{v}_D$ attains 0.

Proof of Equilibrium
Case (a).
For type $\underline{v}_D$:
Case (a.1) In addition to the equilibrium conditions on case (a), assume that $\hat{p}(\underline{v}_D) < p^C$. Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\underline{v}_D)$ weakly dominates all other arming levels. Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. Because high type D's prefer acquiescing to deterring $(0 > \bar{v}_D - K(p^C))$, it implies that low-types also prefer acquiescing to deterring. I can also show that type $\underline{v}_D$ prefers setting $p = p_0$ and acquiescing to fighting. I start with given condition $0 > \bar{U}_D(\hat{p}(\bar{v}_d))$, or

$$0 > -\frac{\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}(nN_D + c_D) + \frac{\alpha\hat{p}(\bar{v}_D)}{\alpha + n\hat{p}(\bar{v}_D)(1-\hat{p}(\bar{v}_D))}\bar{v}_D - K(\hat{p}(\bar{v}_D)).$$

Using that $\hat{p}(\bar{v}_D)$ optimizes the expression on the right and using that $\hat{p}(\underline{v}_D) \in [p^D(\bar{v}_D), p^C]$,[32] I can also say

$$0 > -\frac{\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}{\alpha + n\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}(nN_D + c_D) + \frac{\alpha\hat{p}(\underline{v}_D)}{\alpha + n\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}\bar{v}_D - K(\hat{p}(\underline{v}_D)),$$

and also

$$0 > -\frac{\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}{\alpha + n\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}(nN_D + c_D) + \frac{\alpha\hat{p}(\underline{v}_D)}{\alpha + n\hat{p}(\underline{v}_D)(1-\hat{p}(\underline{v}_D))}\underline{v}_D - K(\hat{p}(\underline{v}_D)),$$

Thus, type $\underline{v}_D$ prefers setting $p = p_0$ and acquiescing to fighting.

Case (a.2) In addition to the equilibrium conditions on case (a), assume that $\hat{p}(\underline{v}_D) \geq p^C$. Within the range $p \in [p_0, p^C)$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. Because $0 > \bar{v}_D - K(p^C)$, it implies that type $\underline{v}_D$ also prefers setting $p = p_0$ and acquiescing to setting $p = p^C$ and deterring.

For type $\bar{v}_D$: Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce,

[32]Because $\hat{p}(\underline{v}_D) \in [p^D(\underline{v}_D), p^C]$, $p^D(\bar{v}_D) < p^D(\underline{v}_D)$, and $\hat{p}(\underline{v}_D) < p^C$.

making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p^C)$, C will challenge and D will fight; thus, in this range $p = \hat{p}(\bar{v}_D)$ weakly dominates all other arming levels.[33] Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $0 > \bar{U}_D(\hat{p}(\bar{v}_d))$ and $0 > \bar{v}_D - K(p^C)$, implying that D prefers selecting p_0 and acquiescing to fighting or deterring .

For C: For $p = p_0$, C believes both types of D are selecting this arming level; thus, C's beliefs on type are the prior distribution of type. For $p \in (p_0, p^C)$, C believes D is a low-type and would acquiesce or fight if challenged, either of which gives C a strictly positive payoff for challenging. For $p \in [p^C, p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (based on the p^C condition).

Case (b). The proof is nearly identical, other than D can no longer select some $p \geq p^C$ and deter C, and C believes that D is a low type for selecting any $p \in (p_0, p_1]$.

3.8 Pooling Equilibrium 2:

- This equilibrium occurs when $\underline{v}_D - K(max\left\{p^D(\bar{v}_D), p_0\right\}) > 0$, $p^C \leq max\left\{p^D(\bar{v}_D), p_0\right\}$,and the set $\tilde{P}$ is non-empty and $\tilde{p} \leq max\left\{p^D(\bar{v}_D), p_0\right\}$,
- Type $\bar{v}_D$ selects $p = max\left\{p^D(\bar{v}_D),\, p_0\right\}$, and type $\underline{v}_D$ selects $p = max\left\{p^D(\bar{v}_D),\, p_0\right\}$.
- C will not challenge when observing $p \geq max\left\{p^D(\bar{v}_D),\, p_0\right\}$. C will challenge when observing $p < max\left\{p^D(\bar{v}_D),\, p_0\right\}$.
- Type $\underline{v}_D$ (who is not challenged) would not escalate if challenged. Type $\bar{v}_D$ (who is not challenged) would escalate if challenged.
- C's Beliefs: If $p = max\left\{p^D(\bar{v}_D),\, p_0\right\}$, then C believes D is low-type with probability $1 - \pi$ and high-type with probability π. If $p < max\left\{p^D(\bar{v}_D),\, p_0\right\}$,[34] then C believes D is low-type with probability 1. If $p > max\left\{p^D(\bar{v}_D),\, p_0\right\}$, then C believes D is high-type with probability 1.
- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(max\left\{p^D(\bar{v}_D),\, p_0\right\})$, type $\underline{v}_D$ attains $\underline{v}_D - K(max\left\{p^D(\bar{v}_D),\, p_0\right\})$.

Proof of Equilibrium

For type $\underline{v}_D$:

Case 1. In addition to the equilibrium conditions, also assume that $p_0 < p^D(\bar{v}_D)$. Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly

[33] If $\hat{p}(\bar{v}_D) = p^C$, then D would optimally select p^C and deter. This prevents open set issues over this range.

[34] Needless to say this belief structure could describe non-feasible actions (when $p_0 > p^D(\bar{v}_D)$). I keep this in place for simplicity.

decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p_1]$, C will not challenge. Thus, in this range, $p = p^D(\bar{v}_D)$ dominates all other arming levels. By assumption $\underline{v}_D - K(p^D(\bar{v}_D)) > 0$, D prefers setting $p = p^D(\bar{v}_D)$ and bluffing to setting $p = p_0$ and acquiescing.[35]

Case 2. Assume $p_0 \geq p^D(\bar{v}_D)$. For all $p \in [p_0, p_1]$, C will not challenge. Thus, in this range, $p = p_0$ dominates all other arming levels.

For type $\bar{v}_D$:
Case 1. In addition to the equilibrium conditions, also assume that $p_0 < p^D(\bar{v}_D)$. Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p_1]$, C will not challenge. Thus, in this range, $p = p^D(\bar{v}_D)$ dominates all other arming levels. By the Parameter Assumptions, $\bar{v}_D - K(p^D(\bar{v}_D)) > 0$, meaning D prefers setting $p = p^D(\bar{v}_D)$ and bluffing to setting $p = p_0$ and acquiescing.

Case 2. In addition to the equilibrium conditions, assume $p_0 \geq p^D(\bar{v}_D)$. For all $p \in [p_0, p_1]$, C will not challenge. Thus, in this range, $p = p_0$ dominates all other arming levels.

For C: For $p = max\left\{p^D(\bar{v}_D),\, p_0\right\}$, C believes both types of D are selecting this arming level; thus, C's beliefs follow the given distribution. For $p \in [p_0, p^D(\bar{v}_D))$ (whenever $p_0 < p^D(\bar{v}_D)$), C believes D is a low-type and would acquiesce if challenged, which gives C a strictly positive payoff for challenging. For $p \in (p^D(\bar{v}_D), p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (because $p^C < p^D(\bar{v}_D)$).

3.9 Pooling Equilibrium 3:

- This equilibrium occurs when the set $\tilde{P}$ is non-empty, $\tilde{p} > max\left\{p^D(\bar{v}_D),\, p_0\right\}$, $\underline{v}_D - K(\tilde{p}) > 0$, and $\tilde{p} < p^D(\underline{v}_D)$.[36]
- Type $\bar{v}_D$ selects $p = \tilde{p}$, and type $\underline{v}_D$ selects $p = \tilde{p}$.
- C will not challenge when observing $p \geq \tilde{p}$ and will challenge when observing $p < \tilde{p}$.
- Type $\underline{v}_D$ (who is not challenged) would not escalate if challenged. Type $\bar{v}_D$ (who is not challenged) would escalate if challenged.
- C's Beliefs: If $p = \tilde{p}$, then C believes D is low-type with probability $1 - \pi$ and high-type with probability π. If $p < \tilde{p}$, then C believes D is low-type with probability 1. If $p > \tilde{p}$, then C believes D is high-type with probability 1.

[35] Because $p^D(\bar{v}_D) < p^D(\underline{v}_D)$, D prefers not fighting.
[36] By transitivity, I can say $\tilde{p} < p_1$.

- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(\tilde{p})$, type $\underline{v}_D$ attains $\underline{v}_D - K(\tilde{p})$.

Proof of Equilibrium

For type $\underline{v}_D$: Within the range $p \in [p_0, \tilde{p})$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [\tilde{p}, p_1]$, C will not challenge. Thus, in this range, $p = \tilde{p}$ dominates all other arming levels. By assumption $\underline{v}_D - K(\tilde{p}) > 0$, meaning D prefers setting $p = \tilde{p}$ C not challenging to setting $p = p_0$ and acquiescing.

For type $\bar{v}_D$: Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), \tilde{p})$, C will challenge and D will fight, meaning D does best selecting some optimal arming level. In the range $p \in [\tilde{p}, p_1]$, C will not challenge. Thus, in this range, $p = \tilde{p}$ dominates all other arming levels. By assumption $\underline{v}_D - K(\tilde{p}) > 0$, implying that high types prefer setting $p = \tilde{p}$ and deterring to setting $p = p_0$ and acquiescing. To demonstrate that high types would never select $p \in [p^D(\bar{v}_D), \tilde{p})$ and fight, I define re-define $\dot{p}$ as

$$\dot{p} \in argmax_{p \in [p^D(\bar{v}_D), \tilde{p}]} \left\{ -\frac{p(1-p)}{\alpha + np(1-p)} (nN_D + c_D) + \frac{\alpha p}{\alpha + np(1-p)} \bar{v}_D - K(p) \right\}.$$

I start using $\underline{v}_D - K(\tilde{p}) > 0$, which is given, and that $\tilde{p} < p^D(\underline{v}_D)$, which implies that low-types would do strictly worse selecting some $\dot{p}$ and fighting relative to setting $p = p_0$ and acquiescing. Using this observation and $\underline{v}_D - K(\tilde{p}) > 0$ (and transitivity) gives

$$\underline{v}_D - K(\tilde{p}) > -\frac{\dot{p}(1-\dot{p})}{\alpha + n\dot{p}(1-\dot{p})} (nN_D + c_D) + \frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \underline{v}_D - K(\dot{p})$$

I can then use that $\bar{v}_D - \underline{v}_D > (\bar{v}_D - \underline{v}_D) \left(\frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \right)$, which gives

$$\bar{v}_D - K(\tilde{p}) > -\frac{\dot{p}(1-\dot{p})}{\alpha + n\dot{p}(1-\dot{p})} (nN_D + c_D) + \frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \bar{v}_D - K(\dot{p}).$$

Thus, high type D's prefer arming to level $\tilde{p}$ and deterring to fighting.

For C: For $p = [p_0, \tilde{p})$, C believes D is a low-type and would acquiesce if challenged, which gives C a strictly positive payoff for challenging. For $p \in (\tilde{p}, p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging. For $p = \tilde{p}$, C's beliefs follow the priors, and C prefers not challenging based on how $\tilde{p}$ is defined.

3.10 Pooling Equilibrium 4:

- This equilibrium occurs when $max\left\{p^D(\underline{v}_D), p_0\right\} \geq p^C$, $\underline{v}_D - K(max\left\{p^D(\underline{v}_D), p_0\right\}) > 0$, and, when the set of $\tilde{P}$ is non-empty, $\tilde{p} \geq max\left\{p^D(\underline{v}_D), p_0\right\}$.
- Type $\bar{v}_D$ selects $p = max\left\{p^D(\underline{v}_D), p_0\right\}$, and type $\underline{v}_D$ selects $p = max\left\{p^D(\underline{v}_D), p_0\right\}$.
- C will not challenge when observing $p = max\left\{p^D(\underline{v}_D), p_0\right\}$, will challenge when observing $p < max\left\{p^D(\underline{v}_D), p_0\right\}$, and will not challenge when observing $p > max\left\{p^D(\underline{v}_D), p_0\right\}$.
- Both types would escalate if challenged.
- C's Beliefs: If $p = max\left\{p^D(\underline{v}_D), p_0\right\}$, then C believes D is low-type with probability $1 - \pi$ and high-type with probability π. If $p < max\left\{p^D(\underline{v}_D), p_0\right\}$ then C believes D is low-type with probability 1. If $p > max\left\{p^D(\underline{v}_D), p_0\right\}$, then C believes D is a high-type with probability 1.
- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(max\left\{p^D(\underline{v}_D), p_0\right\})$, type $\underline{v}_D$ attains $\underline{v}_D - K(max\left\{p^D(\underline{v}_D), p_0\right\})$.

Proof of Equilibrium

For type $\underline{v}_D$:
Case 1. In addition to the equilibrium conditions, assume that $p_0 < p^D(\underline{v}_D)$. Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p_1]$, C will not challenge. Thus, in this range, $p = p^D(\underline{v}_D)$ dominates all other arming levels. By assumption $\underline{v}_D - K(p^D(\underline{v}_D)) > 0$, meaning D prefers setting $p = p^D(\underline{v}_D)$ and deterring to setting $p = p_0$ and acquiescing.

Case 2. In addition to the equilibrium conditions, assume $p_0 \geq p^D(\underline{v}_D)$. For all $p \in [p_0, p_1]$, C will not challenge. Thus, in this range, $p = p_0$ dominates all other arming levels.

For type $\bar{v}_D$:
Case 1. In addition to the equilibrium conditions, also assume that $p_0 < p^D(\underline{v}_D)$. I also assume $p^D(\bar{v}_D) > p_0$; relaxing this makes little difference to the proof, so I will not discuss this alternate case. Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p^D(\underline{v}_D))$, C will challenge and D will fight, selecting some optimal arming level. In the range $p \in [p^D(\underline{v}_D), p_1]$, C will not challenge. Thus, in this range, $p = p^D(\underline{v}_D)$ dominates all other arming levels. By assumption $\underline{v}_D - K(p^D(\underline{v}_D)) > 0$, implying that high types also prefer setting $p = p^D(\underline{v}_D)$ and deterring to setting $p = p_0$ and acquiescing. To demonstrate that high types would never select $p \in [p^D(\bar{v}_D), p^D(\underline{v}_D))$ and fight, I define re-define $\dot{p}$ as

$$\dot{p} \in argmax_{p \in [p^D(\bar{v}_D), p^D(\underline{v}_D)]} \left\{ -\frac{p(1-p)}{\alpha + np(1-p)} (nN_D + c_D) + \frac{\alpha p}{\alpha + np(1-p)} \bar{v}_D - K(p) \right\}.$$

I start using $\underline{v}_D - K(p^D(\underline{v}_D)) > 0$, which is given, and that $\dot{p} < p^D(\underline{v}_D)$, which implies that low-types would do strictly worse selecting some $\dot{p}$ and fighting relative to setting $p = p_0$ and acquiescing.

$$\underline{v}_D - K(p^D(\underline{v}_D)) > -\frac{\dot{p}(1-\dot{p})}{\alpha + n\dot{p}(1-\dot{p})} (nN_D + c_D) + \frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \underline{v}_D - K(\dot{p})$$

I can then use that $\bar{v}_D - \underline{v}_D > (\bar{v}_D - \underline{v}_D) \left(\frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \right)$, which gives

$$\bar{v}_D - K(p^D(\underline{v}_D)) > -\frac{\dot{p}(1-\dot{p})}{\alpha + n\dot{p}(1-\dot{p})} (nN_D + c_D) + \frac{\alpha \dot{p}}{\alpha + n\dot{p}(1-\dot{p})} \bar{v}_D - K(\dot{p}).$$

Thus, high types prefer arming to level $p^D(\underline{v}_D)$ than to fighting.

Case 2. In addition to the equilibrium conditions, assume $p_0 \geq p^D(\underline{v}_D)$. For all $p \in [p_0, p_1]$, C will not challenge. Thus, in this range, $p = p_0$ dominates all other arming levels.

For C: For $p = max \left\{ p^D(\underline{v}_D),\, p_0 \right\}$, C believes both types of D are selecting this arming level; thus, C's beliefs on type are the prior distribution of type. For $p \in [p_0, p^D(\underline{v}_D))$ (whenever $p_0 < p^D(\underline{v}_D)$), C believes D is a low-type and would acquiesce if challenged, which gives C a strictly positive payoff for challenging. For $p \in (p^D(\underline{v}_D), p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (because $p^C < p^D(\underline{v}_D)$).

3.11 Pooling Equilibrium 5

This equilibrium occurs when $\underline{v}_D - K(p^C) > 0$ and $\underline{v}_D - K(p^C) \geq U_D(\hat{p}(\underline{v}_D))$, $max \left\{ p^D(\underline{v}_D), p_0 \right\} < p^C$, $p^C \leq p_1$

- Type $\bar{v}_D$ selects $p = p^C$, and type $\underline{v}_D$ selects $p = p^C$.
- C will not challenge when observing $p \geq p^C$ and will challenge when observing $p < p^C$.
- Both types would escalate if challenged.

- C's Beliefs: If $p = p^C$, then C believes D is low-type with probability $1 - \pi$ and high-type with probability π. If $p < p^C$, then C believes D is low-type with probability 1. If $p > p^C$, then C believes D is high-type with probability 1.
- Payoffs: Type $\bar{v}_D$ attains $\bar{v}_D - K(p^C)$, type $\underline{v}_D$ attains $\underline{v}_D - K(p^C)$.

Proof of Equilibrium

For type $\underline{v}_D$:

Within the range $p \in [p_0, p^D(\underline{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\underline{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\underline{v}_D)$ weakly dominates all other arming levels. Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. By the conditions of the case, $\underline{v}_D - K(p^C) > 0$ and $\underline{v}_D - K(p^C) \geq U_D(\hat{p}(\underline{v}_D))$, implying that D prefers setting $p = p^C$ and deterring to $p = p_0$ and acquiescing or $\hat{p}(\underline{v}_D)$ and fighting.

For type $\bar{v}_D$:

Within the range $p \in [p_0, p^D(\bar{v}_D))$, C will challenge and D will acquiesce, making D's utility strictly decreasing in p. Thus, in this range, $p = p_0$ dominates all other arming levels. Within the range $p \in [p^D(\bar{v}_D), p^C)$, C will challenge and D will fight. Thus, in this range, $p = \hat{p}(\bar{v}_D)$ weakly dominates all other arming levels. Within the range $p \in [p^C, p_1]$, C will not challenge, making D's utility strictly decreasing in p. Thus, in this range, $p = p^C$ dominates all other arming levels. Because $p^D(\underline{v}_D) < p^C$ and $\underline{v}_D - K(p^C) > 0$, the conditions in Lemma 1 hold; thus, because low-types most prefer setting $p = p^C$ and deterring, high types also most prefer this.

For C: For $p = p^C$, C believes both types of D are selecting this arming level; thus, C's beliefs on type are the prior distribution of type. For $p \in [p_0, p^C)$ C believes D is a low-type and would acquiesce if challenged (when $p < p^D(\underline{v}_D)$) or would fight when challenged (when $p \geq p^D(\underline{v}_D)$); in either case, given $p < p^C$, these give C a strictly positive payoff for challenging. For $p \in (p^C, p_1]$, C believes that D is a high type and would fight if challenged, which gives C a weakly negative payoff for challenging (because $p > p^C$).

4 Demonstrating the Equilibrium Satisfies the Intuitive Criterion

For the Pooling 1, Separating 1, Separating 2, Separating 3, Separating 5, and Separating 6 equilibrium spaces, it is straightforward to see high types are doing as good as they can. For example, in Separating 1, high type D's must arm to level $p = max\{p_0, p(\bar{v}_D)\}$ to be willing to fight, and at this level C will not challenge and grant D the asset. If, for example, part of the Separating 1 spaces required D select some $p' > p$ for C to believe D is a high type, then this would not satisfy the intuitive criterion refinement; instead, for all these equilibria, high types D are doing as well as they can in the characterized separating equilibrium (or not separating, in the case of Pooling 1) from low-types.

Pooling 5 also has the feature where high types select the smallest possible value needed to deter C ($p = p^C$). Furthermore, as demonstrated in Lemma 1, high types will always select a weakly greater level of arming than low types; thus the $\underline{v}_D - K(p^C) > 0$ and $\underline{v}_D - K(p^C) \geq U_D(\hat{p}(\underline{v}_D))$ conditions imply that high types will do best selecting p^C over some $p = \hat{p}(\bar{v}_D)$ or $p = p_0$.

It is possible to demonstrate that Pooling 2, Pooling 3, Pooling 4, and Separating 4 all satisfy the intuitive criterion simultaneously. I do this in Lemma 3. To give a sense of what Lemma 3 means, Lemma 3 implies that within Pooling Equilibrium 4, high-type D's will never have an incentive to switch to some p'' where $p^D(\bar{v}_D) \leq p'' < p^D(\underline{v}_D)$ and fight with positive probability relative to arming to $p^D(\underline{v}_D)$ and attaining the asset. Given that Pooling 2-4 and Separating 4 all have the condition where high-types prefer arming to some level $p = p'$ that keeps C from challenging to arming to $p = p_0$ and acquiescing, proving Lemma 3 will imply that the equilibrium above satisfies the intuitive criterion.

Lemma 3: *Suppose an equilibrium exists where C will not challenge upon observing p' where $p = p' \in (p_0, p_1]$, $\underline{v}_D - K(p') \geq 0$, and $p' \leq p^D(\underline{v}_D)$. If for all $p'' \in [p^D(\bar{v}_D), p')$ either (a) C challenges with certainty upon observing p'' or (b) C challenges with probability $1 - \zeta \in (0, 1]$ after observing p'', then high-type D's prefer arming to level p' rather than selecting p'' and fighting with (a) certainty or (b) probability $1 - \zeta$.*

Proof: Any semi-separating equilibrium will take the form of high types arming to level p'' and always fighting when challenged,[37] and low-types mixing between arming to level p_0 and always acquiescing when challenged (where challenging happens with certainty), and arming to level p'' and acquiescing when challenged (where challenging happens with probability $1-\zeta$).[38] For low-type D's to be indifferent between arming to p_0 and always getting challenged, and arming to p'' and getting challenged with probability $1-\zeta$, the following must hold (lest ζ does not support a semi-separating equilibrium):

$$0 = \zeta\,(\underline{v}_D) + (1-\zeta)\,(0) - K(p'').$$

Also note that because $p'' < p^D(\underline{v}_D)$, low-type D's prefer acquiescing to going to war, implying that

[37] High types fight due to $p'' \geq p^D(\bar{v}_D)$

[38] Low types acquiesce because $p'' < p^D(\underline{v}_D)$.

$$0 >\zeta\left(\underline{v}_D\right)+(1-\zeta)\left(-\frac{(nN_D+c_D)p''(1-p'')}{\alpha+np''(1-p'')}+\frac{\alpha}{\alpha+np''(1-p'')}\left(p''\underline{v}_D\right)\right)-K(p'').$$

Because $\underline{v}_D - K(p') \geq 0$, I can say

$$\underline{v}_D - K(p') >\zeta\left(\underline{v}_D\right)+(1-\zeta)\left(-\frac{(nN_D+c_D)p''(1-p'')}{\alpha+np''(1-p'')}+\frac{\alpha}{\alpha+np''(1-p'')}\left(p''\underline{v}_D\right)\right)-K(p'')$$

I add $\bar{v}_D - \underline{v}_D$ to the left-hand-side, and I add $\zeta\left(\bar{v}_D - \underline{v}_D\right) + (1-\zeta)\frac{\alpha p''}{\alpha+np''(1-p'')}\left(\bar{v}_D - \underline{v}_D\right)$ to the right-hand-side. The inequality is preserved because $\frac{\alpha p''}{\alpha+np''(1-p'')} < 1$. This gives

$$\bar{v}_D - K(p') >\zeta\left(\bar{v}_D\right)+(1-\zeta)\left(-\frac{(nN_D+c_D)p''(1-p'')}{\alpha+np''(1-p'')}+\frac{\alpha}{\alpha+np''(1-p'')}\left(p''\bar{v}_D\right)\right)-K(p'').$$

which implies that high-type D's prefer arming to p' and attaining the asset relative to arming to level p'' and fighting over the asset with some probability (as part of the semi-separating equilibrium).

Note that the proof above also functions for the case when C challenges with certainty (set $\zeta = 0$).

5 Remark 1

Remark 1. *High types select greater arming levels (i.e.* $p^*(\bar{v}_D) \geq p^*(\underline{v}_D)$*).*

This follows almost entirely from the discussion of equilibria spaces above. More specifically, for the parameter space where $\underline{v}_D - K(p^D(\underline{v}_D)) < 0$ and $p^D(\underline{v}_D) \geq p^C$, I prove the existence of each equilibrium, and each equilibrium has the property $p^*(\bar{v}_D) \geq p^*(\underline{v}_D)$. For the parameter space where $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$ and $p^D(\underline{v}_D) < p^C$, Lemma 1 implies $p^*(\bar{v}_D) \geq p^*(\underline{v}_D)$.

6 Proof of Remark 2

Remark 2: *Increasing n has ambiguous effects on the equilibrium level of arming* p^**.*

Follows entirely from equilibrium construction.

7 Proving Remark 3

Remark 3 (Nuclear Peace). *Consider nuclear instability parameters* $n', n'' \in \mathbb{R}_+$ *with* $n' < n''$ *. For both* $v_D \in \{\underline{v}_D, \bar{v}_D\}$*, so long that either*
(a) For n'' $p^C \leq p^D(\bar{v}_D)$ *holds, or*

(b) For n'' $p^C > p^D(\bar{v}_D)$ *holds, and for* n'*and* n'' $\hat{U}_D(p, v_D)$ *is concave in* p*,*

then the shift from n' *to* n'' *results in less war.*

Proof. Because this proof is involved, it is worthwhile outlining how I proceed. I begin by discussing "Case 1," which outlines the conditions in Remark 3 (a), where war never happens under n''. I then proceed to the more complex case, "Case 2," which considers the conditions in Remark 3 (b). I first establish a useful lemma, which demonstrates that as n increases, when the conditions in (b) hold, D's utility from war is decreasing. I then establish another useful Lemma, which characterizes the full set of inequalities where low types go to war in equilibrium. For example, one of these inequalities is that low-type D's must do better going to war than acquiesing. I then show the inequalities needed to support the equilibria where low types go to war are strained or break as n increases. Referring back to the example, because low-type D's war utility is decreasing and their "acquiesce" utility is unchanging, the inequality where D prefers fighting to acquiesing is strained or can break. I then repeat the process for high types.

7.1 Case 1: For n'', $p^C \leq p^D(\bar{v}_D)$

Note that $p^D(\bar{v}_D) < p^D(\underline{v}_D)$. Also note that when $p^C \leq p^D(v_D)$, then war is never possible because there is no arming level where C would be willing to challenge and D would be willing to fight.

If for n'' $p^C \leq p^D(\bar{v}_D)$ holds, then for n'' $p^C \leq p^D(\underline{v}_D)$ also holds. This implies that under n'' war will never occur. Therefore, even if n' were such that $p^D(\bar{v}_D) < p^C$ or $p^D(\underline{v}_D) < p^C$ (i.e. war was possible under n'), the likelihood of war would be (weakly) decreasing.

7.2 Case 2: For n'', $p^C > p^D(\bar{v}_D)$ and $\hat{U}_D(p, v_D)$ is concave in p

This proof is assisted by a helpful Lemma that applies to a subset of the parameter space within Case 2. Before introducing it, I need to introduce new notation. I abuse notation and let

$$\hat{U}(p, v_D, n) = -\frac{p(1-p)}{\alpha + np(1-p)}(nN_D + c_D) + \frac{\alpha}{\alpha + np(1-p)}(pv_D) - K(p)$$

denote D's utility for selecting arming level p conditional on D going to war, where D's type is v_D and the nuclear risk is n.

So long that $max\left\{p^D(v_D), p_0\right\} < p^C$ holds[39] for a given n and v_D , $\hat{U}(p, v_D, n)$ can be thought of as the objective function to a constrained optimization. D's full constrained optimization conditional on C challenging and D then going to war is

$$max_{p\in[p_0,p_1]}\hat{U}(p, v_D, n)$$

such that

$$p \geq p^D(v_D)$$

and

$$p^C \geq p,$$

where $p^D(v_D) = 1 - \frac{\alpha v_D}{c_D + nN_D}$ and $p^C = \frac{\alpha v_C}{c_C + nN_C}$ Introducing some (additional) new notation, I let $\hat{V}_D(n)$ denote the value function, or the function that gives the value of the constrained optimization problem at any solution p^* for given type v_D and parameter n (in other words, $\hat{V}_D(n) = max_{p\in[p_0,p_1]}\hat{U}(p, v_D, n)$ such that $p \geq p^D(v_D)$ and $p^C \geq p$).

The helpful Lemma is as follows:

Envelope Setup Lemma: For a fixed $v_D \in \{\underline{v}_D, \bar{v}_D\}$, consider n' and n'' such that $0 < n' < n''$. Assume for n' and n'', $max\left\{p^D(v_D), p_0\right\} < p^C$ and $\hat{U}_D(p, v_D, n)$ is concave in p.[40] $\hat{V}_D(n)$ is decreasing in n.

Proof:

Over the convex compact set $p \in [p_0, p_1]$, K is continuous in p; thus, $\hat{U}(p, v_D, n)$ is continuous in p. Also, as stated in the conditions of the Lemma, $\hat{U}(p, v_D, n)$ is concave in p. It can also be seen that $\hat{U}(p, v_D, n)$, $p^D(v_D)$, and p^C are all differentiable in n, and these derivatives are continuous in p and n. Additionally, by assumption, $\left\{p^D(v_D), p_0\right\} < p^C$, $p^D(v_D) < p_1$, and $p_0 < p_1$, meaning there exist p where the constraints can hold strictly. By Corollary 5 in

[39] Recall I also have $p_0 < p_1$ and $p^D(\underline{v}_D) < p_1$.

[40] This could be further generalized; all that is needed is for $p \in \left[max\left\{p^D(v_D), p_0\right\}, p^C\right]$ $\hat{U}_D(p, v_D, n)$ is concave in p.

Milgrom and Segal (2002), I can say [41]

$$\hat{V}_D(n'') - \hat{V}_D(n') = \int_{n'}^{n''} \left(\frac{(p^*(s)-1)p^*(s)\left(\alpha N_D + \alpha p^*(s)\bar{v}_D - p^*(s)(1-p^*(s))c_D\right)}{\left(-\alpha + s\left(p^*(s)\right)^2 - sp^*(s)\right)^2} \right) ds + \int_{n'}^{n''} \left(\lambda_1 \left(-\frac{\alpha N_D v_D}{\left(c_D + sN_D\right)^2} \right) + \lambda_2 \left(-\frac{\alpha N_C v_C}{\left(c_D + sN_C\right)^2} \right) \right) ds$$

where λ_1 and λ_2 are the nonnegative Lagrange multipliers associated with $p^*(s)$. Note that $(p^*(s)-1) < 0$ and $(\alpha N_D + \alpha p \bar{v}_D - p(1-p)c_D) > 0$,[42] meaning all terms in the integrand are nonpositive. Because the integrand is nonpositive, I can say that $\hat{V}_D(n'') - \hat{V}_D(n') \leq 0$, implying that$\hat{U}(p^*(n), v_D, n)$ is non-increasing in n.

□

I can use the above Lemma to demonstrate that the parameter space where either low-types, high-types, or both declare war is shrinking as n increases. This will rely on the equilibrium characterizations above (Separating 1, Separating 2, etc). I do this in parts, first focusing on showing the low-types will fight less as n increases.

7.2.1 The Parameter Set Where Low Types Fight is Shrinking

I first define the following Lemma:

Lemma: If and only if
(a) when $p^C \leq p_1$, the following conditions hold: $max\left\{p^D(\underline{v}_D), p_0\right\} < p^C$, $\underline{v}_D - K(p^C) < U_D(\hat{p}(\underline{v}_D))$ and $U_D(\hat{p}(\underline{v}_D)) \geq 0$, or

(b) when $p^C > p_1$, the following conditions hold: $U_D(\hat{p}(\underline{v}_D)) \geq 0$,

then low-type D's go to war.

Proof: The "iff" relies on how the conditions above are equivalent to the conditions for Separating equilibria 5 and 6, the only equilibria where low-types go to war. From earlier (see the proof of Separating 5) I can say that $U_D(\hat{p}(\underline{v}_D)) \geq 0$ implies $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$. This both means that low-types will fight, and that Lemma 1 can be applied here. Based on Lemma 1, type $\bar{v}_D$ will either select into fighting (setting $p = \hat{p}(\bar{v}_D)$) or deterring (setting $p = p^C$). When $p^C \leq p_1$, $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$, and the above conditions hold, the equilibrium is Separating 5 (a). When $p^C \leq p_1$, $U_D(\hat{p}(\bar{v}_D)) \leq \bar{v}_D - K(p^C)$, and the above conditions hold, the equilibrium

[41] First order conditions of f are: https://www.wolframalpha.com/input/?i=+%28d%2Fdn+%28-%28p*%281-p%29%28n*N%2Bc%29%29%2Ba*p*v%29%2F%28a%2Bn*p*%281-p%29%29%29

[42] This holds because $p \geq p^D(v_D)$, which implies $0 \leq -n(1-p)N_D + \alpha\bar{v}_D - c(1-p)$.

is Separating 6. And, when $p^C > p_1$ and the above conditions hold, then this is Separating 5 (b).

□

Whenever these constraints hold, low-types will go to war. (and vice versa). From here, I can rely on examining how moving from n' to n'' will alter the constraints. Suppose for n' $p^C \leq p_1$. As n' increases to n'', $p^D(\underline{v}_D)$ is weakly increasing, p_0 is unchanging, and p^C is decreasing, thus making the $max\left\{p^D(\underline{v}_D), p_0\right\} < p^C$ inequality strained (or potentially break). Also as n increases, $\underline{v}_D - K(p^C)$ is increasing , $U_D(\hat{p}(\underline{v}_D))$ is decreasing (as shown above in the Envelope Lemma), and 0 is unchanging, thus making the inequalities $\underline{v}_D - K(p^C) < U_D(\hat{p}(\underline{v}_D))$ and $U_D(\hat{p}(\underline{v}_D)) \geq 0$ strained (or potentially break). Now suppose for n' $p^C > p_1$ holds; through the logic discussed above, the inequalities in this case are strained or could break. Because p^C is decreasing in n, the shift from $n'$to n'' could result in a move from (abusing notation) $p^C(n') > p_1$ to $p^C(n'') \leq p_1$. When this shift occurs, for war to still occur, the additional constraint $\underline{v}_D - K(p^C) < U_D(\hat{p}(\underline{v}_D))$ must also hold; thus, in the shift from n' to n'', all existing constraints become more difficult to satisfy and new constraints must be met, collectively making low-type D's less willing to go to war.

7.2.2 The Parameter Set Where High Types Fight is Shrinking.

Lemma: If and only if

(a) When $p^C \leq p_1$, the following conditions hold: $max\left\{p^D(\bar{v}_D), p_0\right\} < p^C$, $\bar{v}_D - K(p^C) < U_D(\hat{p}(\bar{v}_D))$ and $U_D(\hat{p}(\bar{v}_D)) \geq 0$, or

(b) when $p^C > p_1$, the following conditions hold: $U_D(\hat{p}(\bar{v}_D)) \geq 0$,

then high-type D's go to war.

Proof: Suppose $p^C \leq p_1$. It could also be that
(0) The set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p^C]$ is empty
(1) The set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p^C]$ is non-empty and $U_D(\hat{p}(\underline{v}_D)) \geq 0$; or
(2) The set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p^C]$ is non-empty and $U_D(\hat{p}(\underline{v}_D)) < 0$.

Writing the conditions in (a) with the conditions in (0) and (2) (in other words, fully writing out conditions (0) and (2)) gives:

(0). $p^C \leq p_1$, $max\left\{p^D(\bar{v}_D), p_0\right\} < p^C$, $\bar{v}_D - K(p^C) < U_D(\hat{p}(\bar{v}_D))$ and $U_D(\hat{p}(\bar{v}_D)) \geq 0$ and the set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p^C]$ is empty.

(2). $p^C \leq p_1$, $max\left\{p^D(\bar{v}_D), p_0\right\} < p^C$, $\bar{v}_D - K(p^C) < U_D(\hat{p}(\bar{v}_D))$ and $U_D(\hat{p}(\bar{v}_D)) \geq 0$ and the set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p^C]$ is non-empty and $U_D(\hat{p}(\underline{v}_D)) < 0$.

Together, these are equivalent to Separating 3 (a).

The full set of conditions in (1) are the following: $p^C \leq p_1$, $max\left\{p^D(\bar{v}_D), p_0\right\} < p^C$, $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$, $U_D(\hat{p}(\bar{v}_D)) \geq 0$, the set $[max\left\{p^D(\underline{v}_D), p_0\right\}, p^C]$ is non-empty and $U_D(\hat{p}(\underline{v}_D)) \geq 0$.

These conditions are nearly equivalent to what is stated in Separating 5 (a). At first pass there appears to be two differences, but, as I show below, these difference are effectively ruled out.

First, the conditions for Separating 5 (a) states $max\left\{p^D(\underline{v}_D), p_0\right\} < p^C$, while the conditions on the set in (1) being non-empty imply $max\left\{p^D(\underline{v}_D), p_0\right\} \leq p^C$. In other words, (1) above states it is possible for $max\left\{p^D(\underline{v}_D), p_0\right\} = p^C$, the while Separating 5 (a) conditions do not state this is possible. However, note that the other conditions in (1) imply that this equality can never actually hold. If for high types $max\left\{p^D(\bar{v}_D), p_0\right\} < p^C$, it must be that $p^C > p_0$. Due to this, the remaining distinction between (1) and the conditions in Separating 5 (a) is that (1) also allows for $p^D(\underline{v}_D) = p^C$. However, it cannot ever be the case that $p^D(\underline{v}_D) = p^C$ and $U_D(\hat{p}(\underline{v}_D)) \geq 0$ simultaneously hold when $p_0 < p^C$. Based on how $p^D(\underline{v}_D)$ is defined, the following holds:

$$-\frac{p^D(\underline{v}_D)(1-p^D(\underline{v}_D))}{\alpha + np^D(\underline{v}_D)(1-p^D(\underline{v}_D))}(nN_D + c_D) + \frac{\alpha p^D(\underline{v}_D)}{\alpha + np^D(\underline{v}_D)(1-p^D(\underline{v}_D))}\underline{v}_D = 0.$$

Additionally, because $\hat{p}(\underline{v}_D)$ must fall between $p^D(\underline{v}_D)$ and p^C, when $p^D(\underline{v}_D) = p^C$, it must also be that $\hat{p}(\underline{v}_D) = p^D(\underline{v}_D) = p^C$. Expanding out the expression $U_D(\hat{p}(\underline{v}_D)) \geq 0$ and comparing it to the expression above (note that $U_D(\hat{p}(\underline{v}_D))$ has an additional cost term) gives

$$-\frac{p^D(\underline{v}_D)(1-p^D(\underline{v}_D))}{\alpha + np^D(\underline{v}_D)(1-p^D(\underline{v}_D))}(nN_D + c_D) + \frac{\alpha p^D(\underline{v}_D)}{\alpha + np^D(\underline{v}_D)(1-p^D(\underline{v}_D))}\underline{v}_D - K(p^C) \geq 0.$$

This cannot ever hold: if the top expression equals zero and the bottom expression has a new subtracted cost, then it cannot simultaneously be the case the $\hat{p}(\underline{v}_D) = p^D(\underline{v}_D) = p^C$ and $p_0 < p^C$.

Second, the conditions in Separating 5 (a) does not state that $U_D(\hat{p}(\bar{v}_D)) \geq 0$. However, because $U_D(\hat{p}(\underline{v}_D)) \geq 0$ (which is given in (1)), based on the proof of Separating 5, it implies that $\underline{v}_D - K(p^D(\underline{v}_D)) \geq 0$, meaning Lemma 1 can apply and I know that high-type D's will select a greater arming level. Additionally, $U_D(\hat{p}(\underline{v}_D)) \geq 0$ implies that low-type D's will either fight (set $p = \hat{p}(\underline{v}_D)$) or deter C (set $p = p^C$). Additionally, I know that high-type D's will not set $p = p^C$ due to $U_D(\hat{p}(\bar{v}_D)) > \bar{v}_D - K(p^C)$. Together, this implies that both types of D will fight, meaning $U_D(\hat{p}(\underline{v}_D)) \geq 0$. Thus, the conditions set out in (1) are equivalent to the conditions in Separating 5(a).

Now suppose $p^C > p_1$. It could also be that
(1) $U_D(\hat{p}(\underline{v}_D)) \geq 0$; or
(2) $U_D(\hat{p}(\underline{v}_D)) < 0$.

Writing out conditions (0) and (2) in full gives

(2) $p^C > p_1$, $U_D(\hat{p}(\bar{v}_D)) \geq 0$, and $U_D(\hat{p}(\underline{v}_D)) < 0$.

together, these are the conditions for Separating 3 (b).

Writing out conditions (1) in full gives

(1) $p^C > p_1$, $U_D(\hat{p}(\bar{v}_D)) \geq 0$, and $U_D(\hat{p}(\underline{v}_D)) \geq 0$.

These are the conditions for Separating 5 (b).

Taken with the discussion when $p^C \leq p_1$, I have demonstrated that the conditions in the above Lemma are equivalent to the conditions for Separating 3, Separating 5 (a), and Separating 5 (b), the three settings where high type D's fight.

□

From here, I can examine how moving from n' to n'' will alter the constraints. Suppose for both n' and n'' $p^C \leq p_1$. As n increases, $p^D(\bar{v}_D)$ is weakly increasing, p_0 is unchanging, and p^C is decreasing, thus making the $max\left\{p^D(\bar{v}_D), p_0\right\} < p^C$ inequality strained (or potentially break). Also as n increases, $\bar{v}_D - K(p^C)$ is increasing , $U_D(\hat{p}(\bar{v}_D))$ is decreasing (as shown above), and 0 is unchanging, thus making the inequalities $\bar{v}_D - K(p^C) < U_D(\hat{p}(\bar{v}_D))$ and $U_D(\hat{p}(\bar{v}_D)) \geq 0$ strained (or potentially break). Now suppose for both $n'$and n'' $p^C > p_1$ holds; through the logic discussed above, $U_D(\hat{p}(\bar{v}_D)) \geq 0$ is strained or could break. Because p^C is decreasing in n, the shift from $n'$to n'' could result in a move from $p^C(n') > p_1$ to $p^C(n'') \leq p_1$. When this shift occurs, it imposes additional constraints $max\left\{p^D(\bar{v}_D), p_0\right\} < p^C$, $\bar{v}_D - K(p^C) < U_D(\hat{p}(\bar{v}_D))$ for fighting to still occur; thus, all existing constraints become more difficult to satisfy and new constraints must be met, collectively shrinking the set over which high-type D's go to war.

I have now demonstrated that as n increases, the constraints that result in selection into Separating 3 and Separating 5 all become more difficult to satisfy. Thus, in the shift from n' to n'', the war outcome occurs over a smaller set, or disappears altogether.

8 Remark 4

Remark 4 (Stability-Instability Paradox): *Conditional on D fighting, as n and N_D increase, D will select an arming level that results in a more decisive conflict. Formally, suppose under n and N_D D will select interior arming level $p^* = \hat{p}(v_D)$,*[43] *and that D will go to war.*

(a) If $p^* < \frac{1}{2}$, then $p^*(N_D)$ is decreasing. If $p^* > \frac{1}{2}$, then $p^*(N_D)$ is increasing.

(b) If p^* is small (see Appendix for more details), then $p^*(n)$ is decreasing. If p^* is large, then $p^*(n)$ is increasing.

Proof:

Conditional on D going to war, D's optimization problem is

$$\hat{p}(v_d) \in \underset{p\in[max\{p^D(\bar{v}_D),p_0\},min\{p^C,p_1\}]}{\arg\max} \left\{ -\frac{np(1-p)}{\alpha+np(1-p)}N_D + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - \frac{c_D p(1-p)}{\alpha+np(1-p)} - K(p) \right\}.$$

I once again use Topkis Monotonicity Theory (see Lemma 1). The effects described in Remark 4 are most clear for N_D. Consider the above optimization for N_D and N_D' (with $N_D < N_D'$) and $p, p' \in \left[max\left\{p^D(\bar{v}_D), p_0\right\}, min\left\{p^C, p_1\right\}\right]$ with $p < p'$. For D's optimization problem from going to war to exhibit increasing differences, it must be that

$$-\frac{np'(1-p')}{\alpha+np'(1-p')}N_D' - \left(-\frac{np(1-p)}{\alpha+np(1-p)}N_D'\right) > -\frac{np'(1-p')}{\alpha+np'(1-p')}N_D - \left(-\frac{np(1-p)}{\alpha+np(1-p)}N_D\right)$$

or

$$\frac{np'(1-p')}{\alpha+np'(1-p')}N_D - \frac{np(1-p)}{\alpha+np(1-p)}N_D > \frac{np'(1-p')}{\alpha+np'(1-p')}N_D' - \frac{np(1-p)}{\alpha+np(1-p)}N_D'$$

or more simply

$$(N_D - N_D')\left(\frac{np'(1-p')}{\alpha+np'(1-p')} - \frac{np(1-p)}{\alpha+np(1-p)}\right) > 0.$$

[43] *Formally* $p \in \left(max\left\{p^D(\bar{v}_D), p_0\right\}, min\left\{p^C, p_1\right\}\right)$

The term $N_D - N'_D$ is negative. The expression $\frac{np'(1-p')}{\alpha+np'(1-p')} - \frac{np(1-p)}{\alpha+np(1-p)}$ is (weakly) negative so long that $p' \leq 1/2$.[44] By Topkis Monotonicity Theory, $p^* = \hat{p}(v_D)$ is non-decreasing N_D for all $p^* \geq 1/2$ and non-increasing in N_D for all $p^* \leq 1/2$.

The effects in Remark 4 are less precise for n but still present. While the full increasing-differences expression for n and p is rather complex, it is straightforward to see that in expressing the term, the $K(p)$ and $K(p')$ terms would cancel. Thus, I can turn my attention to properties of the expression $-\frac{np(1-p)}{\alpha+np(1-p)} N_D + \frac{\alpha}{\alpha+np(1-p)}(pv_D) - \frac{c_D p(1-p)}{\alpha+np(1-p)}$, which, unlike K, is twice continuously differentiable, that I will call $\hat{U}_D$. I take the cross partial of $\hat{U}_D$, giving[45]

$$\frac{\partial^2}{\partial p \partial n}\hat{U}_D = \frac{\alpha(2p-1)(\alpha N_D - (1-p)p(2c_D + nN_D)) + \alpha p v_D \left(\alpha(3p-2) - n(1-p)p^2\right)}{\left(\alpha + n(1-p)p\right)^3}$$

This expression is ugly, but consider when $p_0 \approx 0$ and $p_1 \approx 1$. Taking the limits and eliminating terms that obviously go to zero yields:

$$lim_{p\to 0}\left[\frac{\partial^2}{\partial p \partial n}\hat{U}_D\right] = \frac{-(\alpha^2 N_D)}{\alpha^3}$$

$$lim_{p\to 1}\left[\frac{\partial^2}{\partial p \partial n}\hat{U}_D\right] = \frac{\alpha^2 N_D + \alpha^2 v_D}{\alpha^3}$$

While this is not nearly as clean as the N_D expression, but clearly here when p^* is close to zero for a fixed set of parameters then $p^*(n)$ is decreasing; similarly, when p^* is close to 1 for a fixed set of parameters, then $p^*(n)$ is increasing. Once again, there is a push to more decisive conflict by introducing nuclear risk n.

[44] Can be seen by taking the cross partial derivative, or $\frac{\partial^2}{\partial p \partial N_D}\frac{np(1-p)}{\alpha+np(1-p)} = \frac{\alpha n(2p-1)}{(n(p-1)p-\alpha)^2}$.

[45] https://www.wolframalpha.com/input/?i=d%2Fdp+%28d%2Fdn+%28-%28p*%281-p%29%28n*N%2Bc%29%29%2Ba*p*v%29%2F%28a%2Bn*p*%281-p%29%29%29

9 Derivations

9.1 Deriving Arming Levels

I first derive p^C, the conventional force level that would would make C indifferent between challenging or not, conditional on D escalating in stage 4

$$0 = -\frac{n}{h}N_C + \frac{\alpha}{hp^C(1-p^C)}\big((1-p^C)v_C\big) - \frac{c_C}{h}$$

$$0 = -np^C N_C + \alpha v_C - c_C p^C$$

$$p^C = \frac{\alpha v_C}{c_C + nN_C}$$

Next I derive $\bar{p}^D$ as the conventional force level that would make a $\bar{v}_D$ D willing to escalate conditional on C challenging.

$$0 \leq \frac{n}{h} * (-N_D) + \frac{\alpha}{hp^D(1-p^D)}\big(p^D \bar{v}_D\big) - \frac{c}{h}$$

$$0 \leq -n(1-p^D)N_D + \alpha\bar{v}_D - c(1-p^D)$$

$$n(1-p^D)N_D + c(1-p^D) \leq \alpha\bar{v}_D$$

$$p^D \geq 1 - \frac{\alpha\bar{v}_D}{c_D + nN_D}.$$

Next, I derive $\tilde{p}$. I use $h(p) = \frac{\alpha + np(1-p)}{p(1-p)}$

$$0 = \pi\left(-\frac{n}{h(\tilde{p})}N_C + \frac{\alpha}{h(\tilde{p})\tilde{p}(1-\tilde{p})}\left((1-\tilde{p})v_C\right) - \frac{c_C}{h(\tilde{p})}\right) + (1-\pi)v_C,$$

$$0 = \pi\left(-nN_C + \frac{\alpha}{\tilde{p}(1-\tilde{p})}\left((1-\tilde{p})v_C\right) - c_C\right) + (1-\pi)v_C\left(\frac{\alpha + n\tilde{p}(1-\tilde{p})}{\tilde{p}(1-\tilde{p})}\right),$$

$$0 = \pi\left(-nN_C\tilde{p}(1-\tilde{p}) + \left((1-\tilde{p})\alpha v_C\right) - c_C\tilde{p}(1-\tilde{p})\right) + (1-\pi)v_C\left(\alpha + n\tilde{p}(1-\tilde{p})\right),$$

$$0 = \tilde{p}(1-\tilde{p})\left(-\pi n N_C - \pi c_C + (1-\pi)v_C n\right) - \tilde{p}\alpha\pi v_C + \alpha\pi v_C + (1-\pi)\alpha v_C$$

$$0 = -\tilde{p}^2\left(-\pi n N_C - \pi c_C + (1-\pi)v_C n\right) + \tilde{p}\left(-\pi n N_C - \pi c_C + (1-\pi)v_C n - \alpha\pi v_C\right) + \pi\alpha v_C + (1-\pi)\alpha v_C$$

I can solve for $\tilde{p}$ using the quadratic formula. Letting $a = -\left(-\pi n N_C - \pi c_C + (1-\pi)v_C n\right)$, $b = \left(-\pi n N_C - \pi c_C + (1-\pi)v_C n - \alpha\pi v_C\right)$, and $c = \alpha v_C$, this is simply

$$\tilde{p} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Or

$$\tilde{p} = \frac{-\left(-\pi n N_C - \pi c_C + (1-\pi)v_C n - \alpha\pi v_C\right) \pm \sqrt{\left(-\pi n N_C - \pi c_C + (1-\pi)v_C n - \alpha\pi v_C\right)^2 + 4\left(-\pi n N_C - \pi c\right.}}{-2\left(-\pi n N_C - \pi c_C + (1-\pi)v_C n\right)}$$

Alternatively, I can try the following. I know $h(p) = \frac{\alpha + np(1-p)}{p(1-p)}$

$$0 = \pi\left(-\frac{n}{h(\tilde{p})}N_C + \frac{\alpha}{h(\tilde{p})\tilde{p}(1-\tilde{p})}\left((1-\tilde{p})v_C\right) - \frac{c_C}{h(\tilde{p})}\right) + (1-\pi)v_C,$$

or

$$v_C = \left(\frac{\pi n}{h(\tilde{p})}N_C + \frac{\pi c_C}{h(\tilde{p})}\right)\left(\frac{\pi\alpha}{h(\tilde{p})\tilde{p}} + 1 - \pi\right)^{-1}.$$

or

$$v_C = \left(\pi n N_C + \pi c_C\right)\left(\frac{\pi\alpha}{\tilde{p}} + \frac{(1-\pi)(\alpha + n\tilde{p}(1-\tilde{p}))}{\tilde{p}(1-\tilde{p})}\right)^{-1}.$$

10 Scratch for Graphing/Derivations for Matlab

10.0.1 Separating 1: Matlab calculations

East boundary:

$$\underline{v}_D - K(p^D(\bar{v}_D)) < 0$$

$$\underline{v}_D - k(p^D(\bar{v}_D) - p_0)^2 < 0$$

$$\underline{v}_D < k(p^D(\bar{v}_D) - p_0)^2$$

North boundary, $p_D(\bar{v}_D) = p_C$.

$$p_D(\bar{v}_D) = \frac{\alpha v_C}{c_C + nN_C}$$

$$\frac{p_D(\bar{v}_D)(c_C + nN_C)}{\alpha} = v_C$$

10.0.2 Separating 2 Matlab

Estimating the upper bound is the new part here. In S2, high D's select p^C, low D's select p_0. At the upper bound, either (a) high D's decide it is no longer worth it to arm to level p^C or are not able to arm to p^C. At this point, they can either arm knowing they will be fighting, or select p_0. I consider both.

First, it could be that D is solving the optimization

$$p \in max < \left\{ \frac{n}{h(p)} * (-N_D) + \frac{\alpha}{h(p)p(1-p)} (pv_D) - \frac{c_D}{h(p)} - K(p) \right\}$$

or

$$\hat{p}(v_d) \in \underset{p \in [p^D, p^C]}{\arg\max} < \left\{ -\frac{np(1-p)}{\alpha + np(1-p)} N_D + \frac{\alpha}{\alpha + np(1-p)} (pv_D) - \frac{c_D p(1-p)}{\alpha + np(1-p)} - k(p-p_0)^2 \right\}$$

I then use the optimized utility value $U_D(\hat{p}(\bar{v}_D)$ and calculate

$$\bar{v}_D - U_D(\hat{p}(\bar{v}_D) = k * (p^C - p_0)^2$$

$$\left(\frac{\bar{v}_D - U_D(\hat{p}(\bar{v}_D)}{k}\right)^{1/2} = p^C - p_0$$

$$\left(\frac{\bar{v}_D - U_D(\hat{p}(\bar{v}_D)}{k}\right)^{1/2} + p_0 = \frac{\alpha v_C}{c_C + nN_C}$$

$$\frac{c_C + nN_C}{\alpha}\left(\left(\frac{\bar{v}_D - U_D(\hat{p}(\bar{v}_D)}{k}\right)^{1/2} + p_0\right) = v_C$$

Alternatively, the upper bound on this could simply be

$$\frac{\bar{v}_D}{k} = (p^C - p_0)^2$$

$$\frac{\bar{v}_D}{k} = \left(\frac{\alpha v_C}{c_C + nN_C} - p_0\right)^2$$

$$\sqrt{\frac{\bar{v}_D}{k}} = \frac{\alpha v_C}{c_C + nN_C} - p_0$$

$$\frac{(c_C + nN_C)\left(\sqrt{\frac{\bar{v}_D}{k}} + p_0\right)}{\alpha} = v_C$$

Working through that southeast line too: I have the equation $\underline{v}_D - K(p_C) = 0$. I take this and say

$$k(p^C - p_0)^2 = \underline{v}_D$$

$$p^C = \left(\frac{\underline{v}_D}{k}\right)^{1/2} + p_0$$

$$\frac{\alpha v_C}{c_C + nN_C} = \left(\left(\frac{\underline{v}_D}{k}\right)^{1/2} + p_0\right)$$

$$v_C = \frac{c_C + nN_C}{\alpha}\left(\left(\frac{\underline{v}_D}{k}\right)^{1/2} + p_0\right)$$

10.0.3 Separating 4 Matlab

This is the north line.

$$\underline{v}_D - K(p^c) = 0$$

$$\underline{v}_D - k\left(\frac{\alpha v_C}{c_C + nN_C} - p_0\right)^2 = 0$$

$$\frac{\underline{v}_D}{k} = \left(\frac{\alpha v_C}{c_C + nN_C} - p_0\right)^2$$

$$\sqrt{\frac{\underline{v}_D}{k}} = \frac{\alpha v_C}{c_C + nN_C} - p_0$$

$$\frac{(c_C + nN_C)\sqrt{\frac{\underline{v}_D}{k}} + p_0}{\alpha} = v_C$$

For the right boundary

$$\underline{v}_D - k(p^D(\underline{v}_D)) = 0$$

$$\underline{v}_D - k((1-p_0) - \frac{\alpha \underline{v}_D}{c_D + nN_D})^2 = 0$$

$$\underline{v}_D - k(1-p_0)^2 + (2-2p_0)k\frac{\alpha \underline{v}_D}{c_D + nN_D} - k\frac{\alpha^2 \underline{v}_D^2}{(c_D + nN_D)^2} = 0$$

$$-\underline{v}_D^2\left(\frac{k\alpha^2}{(c_D + nN_D)^2}\right) + \underline{v}_D\left(1 + (2-2p_0)\frac{\alpha k}{c_D + nN_D}\right) - k(1-p_0)^2 = 0$$

$$\underline{v}_D^2\left(\frac{k\alpha^2}{(c_D + nN_D)^2}\right) - \underline{v}_D\left(1 + (2-2p_0)\frac{\alpha k}{c_D + nN_D}\right) + k(1-p_0)^2 = 0$$

This leaves me with the following:

$$\underline{v}_D = \frac{\left(1 + (2-2p_0)\frac{\alpha k}{c_D + nN_D}\right) \pm \sqrt{\left(1 + (2-2p_0)\frac{\alpha k}{c_D + nN_D}\right)^2 - 4\left(\frac{k\alpha^2}{(c_D + nN_D)^2}\right)k(1-p_0)^2}}{2\left(\frac{k\alpha^2}{(c_D + nN_D)^2}\right)}$$

10.0.4 Pooling 2 Matlab

I must solve for the north boundary—i.e. the v_C value where $p^D(\bar{v}_D) = \tilde{p}$. I do the following:Alternatively, I can try the following. I know $h(p) = \frac{\alpha + np(1-p)}{p(1-p)}$

$$0 = \pi\left(-\frac{n}{h(\tilde{p})}N_C + \frac{\alpha}{h(\tilde{p})\tilde{p}(1-\tilde{p})}((1-\tilde{p})v_C) - \frac{c_C}{h(\tilde{p})}\right) + (1-\pi)v_C,$$

or

$$v_C = \left(\frac{\pi n}{h(\tilde{p})}N_C + \frac{\pi c_C}{h(\tilde{p})}\right)\left(\frac{\pi\alpha}{h(\tilde{p})\tilde{p}} + 1 - \pi\right)^{-1}.$$

Multiply top and bottom of RHS by $h(\tilde{p})$ to give

$$v_C = (\pi n N_C + \pi c_C)\left(\frac{\pi\alpha}{\tilde{p}} + \frac{(1-\pi)(\alpha + n\tilde{p}(1-\tilde{p}))}{\tilde{p}(1-\tilde{p})}\right)^{-1}.$$

I can then substitute in the value of $p^D(\bar{v}_D)$.

10.0.5 Pooling 3 Matlab

The tricky part here is the north region. Here is how I do it to avoid buggy optimization. For the range of feasible $\underline{v}_D$, I calculate the $p^D(\underline{v}_D)$'s that can be supported. This is easy, but this should be treated as a temporary value (hence subscript).

$$\tilde{p}_t = \sqrt{\frac{\underline{v}_D}{k}} + p_0$$

I then calculate the v_C's that could be supported. This is solve for using

$$v_{C,t} = (\pi n N_C + \pi c_C)\left(\frac{\pi\alpha}{\tilde{p}_t} + \frac{(1-\pi)(\alpha + n\tilde{p}_t(1-\tilde{p}_t))}{\tilde{p}_t(1-\tilde{p}_t)}\right)^{-1}.$$

However, I can't stop there. $v_{C,t}$ is non-monotonic in $\tilde{p}_t$. I resolve this in the code by finding the point the increases in $\tilde{p}_t$ cause $v_{C,t}$ to start decreasing, then "cap" it.

Then from the other side. I also need this to hold: $\tilde{p} < p^D(\underline{v}_D)$, because if not, then the above $\tilde{p}$ is not relevant. So I use

$$p^D(\underline{v}_D) = 1 - \frac{\alpha \underline{v}_D}{c_D + nN_D}$$

to estimate each $p^D(\underline{v}_D)$ for a fixed $\underline{v}_D$. When $p^D(\underline{v}_D)$ is less than $\tilde{p}$, then there is a problem with what I did above. So, I use that new (smaller) $p^D(\underline{v}_D)$ to see what v_C's will be deterred from that by plugging it into the tilde expression for $v_{C,t}$.

10.0.6 Pooling 4 Matlab

Need to derive north border.

For Pooling 4: The northern boundary is based on $p^D(\underline{v}_D) = p^C$. So I calculate the following:

$$1 - \frac{\alpha \underline{v}_D}{c_D + nN_D} = \frac{\alpha v_C}{c_C + nN_C}$$

$$v_C = \frac{c_C + nN_C}{\alpha} \left(1 - \frac{\alpha \underline{v}_D}{c_D + nN_D}\right)$$

10.0.7 Pooling 5 Matlab

$$\underline{v}_D - K(p^c) = 0$$

$$\underline{v}_D - k\left(\frac{\alpha v_C}{c_C + nN_C} - p_0\right)^2 = 0$$

$$\frac{\underline{v}_D}{k} = \left(\frac{\alpha v_C}{c_C + nN_C} - p_0\right)^2$$

$$\sqrt{\frac{\underline{v}_D}{k}} = \frac{\alpha v_C}{c_C + nN_C} - p_0$$

$$\frac{(c_C + nN_C)\left(\sqrt{\frac{\underline{v}_D}{k}} + p_0\right)}{\alpha} = v_C$$

I could also need

$$p^C = p_1$$

$$\frac{\alpha v_C}{c_C + nN_C} = p_1$$

$$\frac{p_1(c_C + nN_C)}{\alpha} = v_C$$